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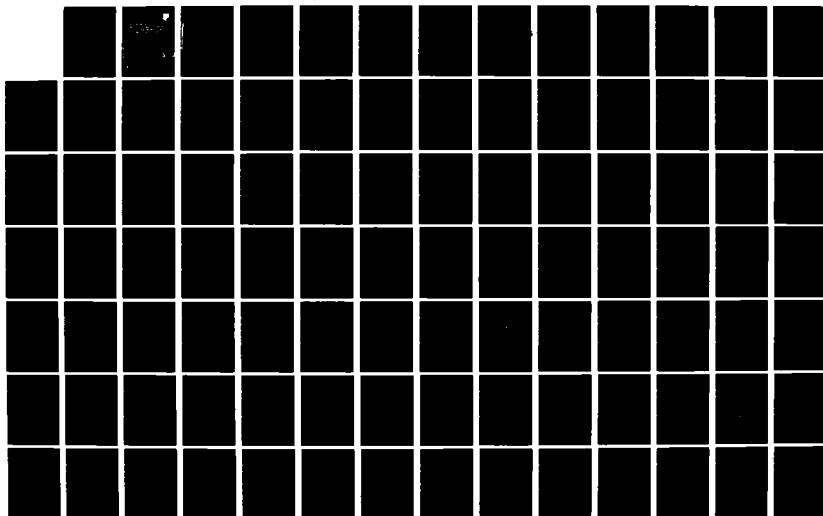PERTURBED BY OSCILLATOR P. (U) ROME AIR DEVELOPMENT
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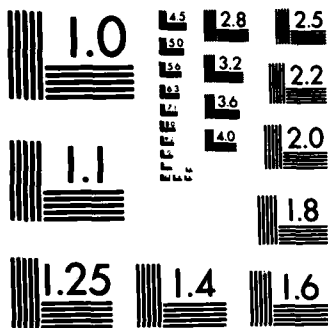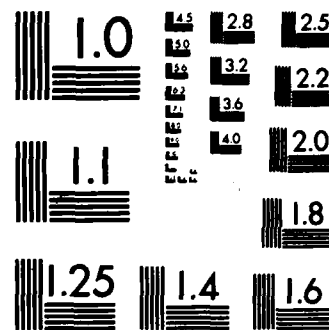UNCLASSIFIED   RADC-TR-82-170   F/G 17/9   NL

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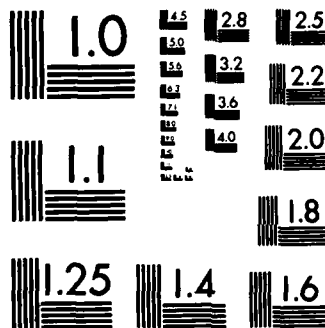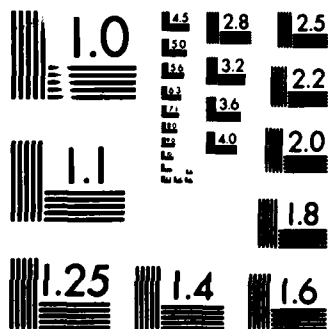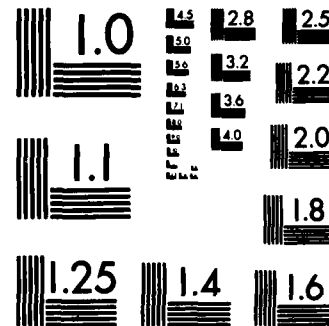
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MODELING AND PROPERTIES OF MODULATED RF SIGNALS PERTURBED BY OSCILLATOR PHASE INSTABILITIES AND RESULTING SPECTRAL DISPERSION

Vincent C. Vannicola

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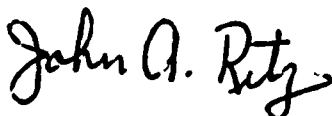
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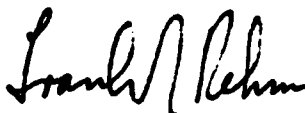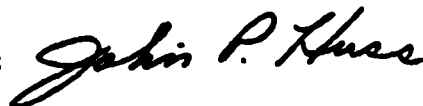| REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

This dissertation considers the problem of characterization and stochastic properties of oscillator phase instabilities. Spectral dispersion caused by such instabilities is also determined. The instabilities are modeled using white phase, random walk phase, random walk frequency random processes as well as drift phase and frequency terms. Auto-correlation functions and power spectral densities are derived for the oscillator rf output signal containing such instabilities. Furthermore, a procedure is proposed which

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involves the covariance matrix of the phase random process in conjunction with the characteristic function for determining the higher order moments of the rf signal. Stationarity and ergodicity properties of the rf signals containing phase instabilities are discussed. In each case, conditions for ergodicity are established for the mean, auto-correlation function, and the power spectral density.

The spectral dispersion generated by oscillator phase instabilities imposes limits on dynamic range and Doppler resolution in radar signal processing. This spectral dispersion is determined for the phase instability models considered in this dissertation. Modulating waveforms considered as examples include the cw, the infinite pulse train and the finite pulse train. Exact closed form expressions are derived and plotted. For low level phase instabilities, first order approximations to the exact expressions are obtained and plotted.

PREFACE

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ILLUSTRATIONS

I INTRODUCTION

1.1. Importance and Problem Definition

Precision frequency sources such as quartz-crystal oscillators and atomic frequency standards and clocks find a wide variety of applications in many engineering fields [1.1],[1.2]. Some important application areas are doppler radar systems; satellite and spacecraft for guidance, tracking and communications; range measurements and digital communication systems. In these and other coherent information processing systems, stability of the precision frequency sources is of prime importance in maintaining synchronization. Most practical sources, however, exhibit some degree of frequency instability. This may be due to internal or external environmental conditions which the frequency source is exposed such as additive and multiplicative noise, temperature fluctuations, supply voltage variations, magnetic field, atmospheric pressure, humidity and vibration. These and other time dependent physical causes generate random time-varying fluctuations and instabilities in the parameters which characterize the oscillator, viz., frequency and phase, which result in a degradation in the performance of the coherent information processing systems. In order to evaluate the actual system performance, an accurate characterization of the instability and its description in terms of some quantitative measures is essential [1.3], [1.4]. By phase/frequency instability, we mean the degree to which an oscillator produces random unwanted and time dependent departures from a pure sine wave. In the spectrum representation, it is the degree to which these departures cause the oscillator specular component to disperse. The consequences of instability are losses in temporal synchronization and spectral

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resolution in the associated system. In this dissertation, we generally model the phase error process as power law spectral density models [1.3], i.e., power spectral density is modeled as a polynomial in the frequency domain. These models tend to have certain physical origins, [1.4], as follows:

a) White phase noise, $y(u)$, is usually due to white noise sources external to the oscillator loop.

b) Random walk phase (white frequency) noise, $x(u)$, usually arises from additive white noise internal to the oscillator loop.

c) Random walk frequency noise, $v(u)$, is usually related to the oscillator environment such as temperature, vibration and shocks.

The objective here is to characterize these undesired time-dependent frequency departures and then utilize it for the evaluation of their effects on the performance of coherent information processing systems.

Since any frequency variation implies an associated phase variation, we shall concentrate on the phase random process as the source of frequency instability and develop stochastic models for it. Although the model and the related discussion is valid for numerous applications, emphasis will be placed on radar/communication applications. In these applications, frequency instability is just one of the many sources of system performance degradation. Frequency instability induces a spectral spread which, in turn, affects the probability of detection of the radar/communication system. This

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dissertation treats only the effects of phase/frequency instability on the system performance. Other sources of system performance degradation such as additive and multiplicative noise; doppler spread of desired and undesired scatterers; atmospheric chatacteristics; spurious modulations and nonlinearities from other components; channel mismatch and fluctuations etc. are not discussed here. They have been discussed in detail elsewhere, e.g. [1.5]-[1.8].

1.2. Previous Work

As indicated above, characterization of frequency instability is an important problem and has received considerable attention. This problem was addressed extensively in a special issue of the Proceedings of the IEEE [1.2]. Throughout that issue, frequency instability characterization in both frequency and time domains and translations between them were considered. Topics in frequency control, frequency stability requirements in various fields of applications, and progress in atomic and quartz frequency control were discussed. That issue was an outgrowth of the tremendous effort put forth in the United States space program to satisfy the urgent need to standardize terminology. It was a first step toward bringing together, in a coordinated way, the latest information related to this field including a number of papers dealing with the fundamentals. There still remained individual preferences for terminology and notation. Three classes of papers constituted the issue: papers dealing with theroetical issues, papers providing the details of oscillators, and papers describing the requirements and specifications from the user's point of view.

Cutler and Searle [1.9] discussed sources of quartz oscillator

noise and associated frequency fluctuation with respect to averaging time. Atomic standards, both passive and active, using servo-controlled oscillation, were discussed with respect to servo time constants. Conventional methods for handling random signals were applied to phase and frequency fluctuations for obtaining meaningful criteria for specifying oscillator stability.

Vessot et al [1.10] described an auto-correlation technique for measuring the short-term properties of stable oscillators. The phase and frequency were converted to amplitude fluctuations via a function multiplier, a second oscillator, and a time averager. Mean square averages of frequency phase, and amplitude were obtained from measuremants on hydrogen masers.

Allan [1.11] developed the theoretical relationship for determination of frequency standard deviation for any finite number of data samples with respect to the infinite time average. He also provided a method for determining power spectral density which is system dependent, i.e. it depends on bandwidth and sampling time. Such diverse topics as flicker noise, masers, the cesium beam standard, and rubidium gas cells were also discussed. These topics were further addressed in [1.3]. A measure of frequency stability was defined with respect to the spectral density of fractional frequency fluctuations. An alternate measure was also given; the expected variance for the N sample averages of the fluctuations taken over some time duration, t, with the begining of successive periods spaced every T units of time. The variance was denoted by

$$\sigma_y^2(N,T,t)$$

A preferred measure of frequency stability in the time domain was achieved by choosing N = 2 and T = t to obtain the "Allan variance", which found widespread use in situations where system noise bandwidth enters into the measurements. Systematic variations, frequency and time domain relationships, applications, and measurement techniques were also treated.

An excellent survey of the state-of-the-art in this area was presented in [1.4]. Many parameters proposed for phase and frequency instability characterization were reviewed including both the widely used classical concepts and the more recent less familiar approaches. They included those recommended by the IEEE Subcommittee on Frequency Stability and by Study Group 7 on "Standard Frequencies and Time Signals" of the International Radio Consultative Committee (CCIR). The paper treated transformations of the error phase and frequency functions between time and spectral domains and the advantages of new approaches that provide improved understanding. Here, the structure functions of phase and its relations with the sample variance and the Hadamard variance were discussed. Polynomial phase and frequency drifts along with random processes were modeled by power-law spectral densities.

Specific frequency stability issues related to radar have been presented by Raven [1.12] who derived oscillator requirements for coherent radars. The modulation effects imposed by phase instabilities within the oscillator were examined for their effects on the radar waveform and system performance. Leeson and Johnson [1.13] treated short-term frequency stability along with system and circuit requirements for operation of Doppler radars in severe vibration and acoustic environments. Linewidth and spectrum are used to define the short-term stability of a Doppler radar.

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Lindsey and Meyr [1.14] determined the time dependent probability density function of the phase error as well as the distribution of cycle clips using a Wiener (Brownian Motion) process embedded in a renewal process of a first order correlative tracking system with periodic nonlinearity. The relation between phase and instantaneous frequency was studied by Boileau and Picinbono [1.15]. In particular, conditions under which the phase cannot be stationary or the instantaneous frequency does not exist were discussed. A new method of analysis, the method of finite-time frequency control to describe the frequency stability was compared with the Allan-Variance procedure.

The Allan variance was used by Yoshimura [1.16] to estimate the variance of several simulated phase random processes containing frequency noise. Distributions of the estimates were compared to the chi-square distribution. It was found that the auto-correlation function of the process affects the outcome for white, flicker, and random walk phase noise. For flicker and random walk frequency, the process can be regarded as nearly independent for a reasonable estimate of the variance.

In several papers, Lindsey and Lewis [1.17] and Lindsey and Chie [1.18] used structure functions in characterizing and relating phase instability with time and frequency domain measures. They demonstrated that the fractional frequency deviation introduced by Cutler and Searle is related to the first order phase structure function. Higher order structure functions in phase and frequency were also treated for their usefulness in overcomming problems associated with frequency drift and flicker type noise convergence.

In this dissertation, a covariance matrix approach for the characterization of phase instability is developed. This allows a uniform and systematic treatment of all the above models and their properties which include those of Lindsey's and Chie's structure functions. The approach provides a geometric mathematical framework which allows for the concise establishment and organization of the stochastic operations required for the treatment of auto-correlation functions, characteristic functions, and power spectral densities of oscillator rf signals having phase instabilities. We identify a class of non-stationary phase random processes which allow determination of the rf signal power spectral density. We also show that under certain conditions on these processes the mean, auto-correlation function, and power spectral density of the rf signal can be determined. Stationarity and ergodicity conditions for the rf signal are given through a number of theorems. Also, since we are interested in communications and radar applications, we consider the effects which the oscillator phase instabilities have on modulation with envelope waveforms. The total effects due to phase instability and envelope modulation are illustrated through numerous examples.

1.3. Analytical Background and Approach

To determine the spectral spreading of a radar or communications waveform as affected by oscillator phase instabilities we call to mind a number of analytical relationships. We make use of the fact that the spectral characterization involves the Fourier transform of the auto-correlation function or average auto-correlation function, $Rf(u)$, of the waveform, $f(u)$. Such a waveform is the product of the system's

deterministic modulating envelope waveform, fo(u), and the oscillator rf

signal, b(u), which in general contains the stationary and

non-stationary phase instabilities, z(u). It will be shown later that

b(u) is stationary in auto-correlation function for a class of

non-stationary random processes, z(u). By the oscillator rf signal,

b(u), we mean [1.17] the sinusoidal function ^{1} of the phase random

process, say z(u), which is a function of time, u. Hence

$$b(u) = A \exp\{ \; j[\phi + z(u)] \; \} \qquad\qquad (1.1)$$

where A and ϕ are arbitrary constants. Modeling of b(u) is considered

in detail in chapter two. The random process, b(u), may contain the

effects of short, intermediate and long term oscillator phase

instabilities as well as other non-deterministic modulations induced by

propagation effects, thermal drift, and doppler spreading of scatterers.

In this dissertation, we will concentrate only on the effects of

oscillator phase instabilities. System average power/energy spectrum

will be determined for a variety of waveforms affected by a set of phase

instability parameters using the procedures as described below.

Let us denote by $f_k(u)$, a sample function of the modulated signal

obtained when we multiply the modulating envelope waveform ^{2}, $f_o(u)$,

---

1.
    The function indicated in equation (1.1) is also known as the cisoidal function. In this dissertation, the term, "sinusoidal", is used to describe this function.

2.
    The tilda (~) over the function denotes complex low pass equivalent. For notational simplicity, this symbol will be omitted with the assumption that the real nature of the phase random process, z(u), and the complex nature of the rf waveforms, f(u) and b(u), is implied throughout this dissertation.

(which in general is non-stationary) by a sample function, $b_k(u)$, (which is a stationary random process, $b(u)$) i.e.,

$$f_k(u) = f_o(u) b_k(u) \qquad (1.2)$$

Figure 1.1 shows the equivalent block diagram of such a modulation operation. It has been shown [1.19], [1.20] that the time auto-correlation function of this product may be determined as follows. First we write the time average auto-correlation function, $R_{fk}(t)$, of the modulated signal. For a power limited waveform, we write

$$R_{fk}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_k(u) f_k^*(u-t) \, du$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} b_k(u) b_k^*(u-t) f_o(u) f_o^*(u-t) \, du \qquad (1.3a).$$

and for an energy limited waveform we write$^{\underline{1}}$,

$$R_{fk}(t) = \int_{-\infty}^{\infty} f_k(u) f_k^*(u-t) \, du$$

$$= \int_{-\infty}^{\infty} b_k(u) b_k^*(u-t) f_o(u) f_o^*(u-t) \, du \qquad (1.3b).$$

$\underline{1}$ We will encounter both mean square power limited and mean square energy limited modulated waveforms. Without loss in generality we will, in this chapter, adhere to the notation for the mean square energy limited case thereby omitting division by 2T and the associated limiting operation.

(a)

OUTPUT WAVEFORM OF MODULATOR

(b)

OUTPUT CORRELATION FUNCTION OF MODULATOR
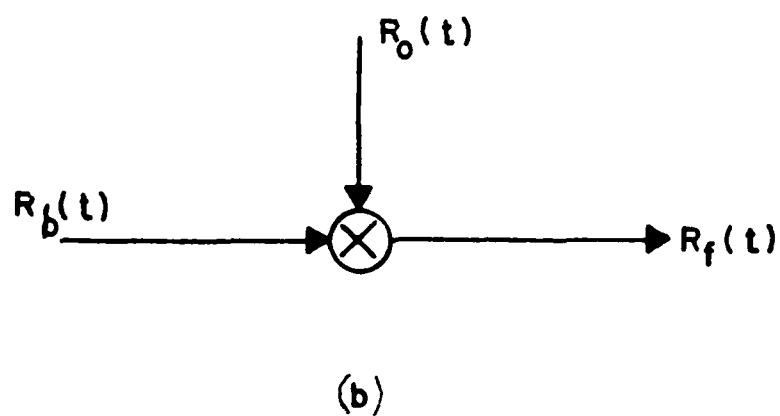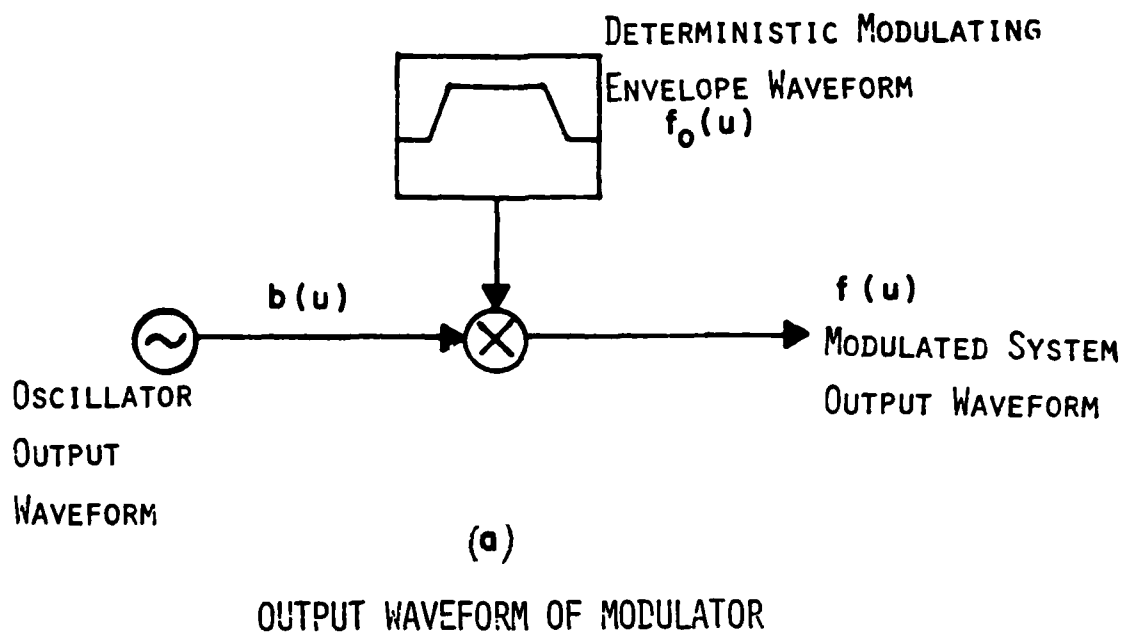
FIGURE 1.1. MODULATOR BLOCK DIAGRAM

The Fourier transform of $R_{fk}(t)$ in equation (1.3a) and equation (1.3b)
is the power spectrum and energy spectrum of the waveform respectively.
These functions are complex lo. pass equivalents and * denotes complex
conjugate. Because the time domain process, $f_o(u)$, and likewise, $f_k(u)$
defined in equation (1.2), is generally not stationary, $R_{fk}(t)$ defined
in equation (1.3) is not, in general, equal to the ensemble average
auto-correlation function, i.e.

$$R_{fk}(t) \neq E \{ f_k(u) \, f_k(u-t) \}$$

Therefore, $R_{fk}(t)$ is not the inverse transform of a power spectral
density. In fact, $R_{fk}(t)$ obtained through equation (1.3) will be
different for different sample functions, $b_k(u)$, and thus it is a sample
function of a random process itself. Since $R_{fk}(t)$ is a sample function
from a random process, we can determine the expected value of $R_{fk}(t)$,
(likewise the Fourier transform, $S_{fk}(w)$) by taking the ensemble average.

$$R_f(t) = E_k \{ R_{fk}(t) \}$$

$$= E_k \{ \int_{-\infty}^{\infty} b_k(u) \, b_k(u-t) \, f_o^*(u) \, f_o^*(u-t) \, du \}$$

Interchanging the order of averaging and integration we obtain

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$$R_f(t) = \int_{-\infty}^{\infty} E_k\{b_k(u)\, b_k(u-t)\}\, f_o^*(u)\, f_o^*(u-t)\, du$$

$$= \int_{-\infty}^{\infty} R_b(t)\, f_o(u)\, f_o^*(u-t)\, du$$

The result is the average auto-correlation function[1] of f(u) [1.20], section 12.3.

$$R_f(t) = R_b(t)\, R_o(t) \tag{1.4}$$

where $R_o(t) \underset{=}{\Delta} \int_{-\infty}^{\infty} f_o(u)\, f_o^*(u-t)\, du$ is the time average

auto-correlation function of the modulating envelope waveform, $f_o(u)$, and $R_b(t)$ is the ensemble average auto-correlation function of the process, b(u). Therefore, the expected value of the time average sample auto-correlation function, $R_{fk}(t)$, i.e. the average auto-correlation function is the product of the stationary process auto-correlation function and the time average envelope auto-correlation function. Upon modulation, the carrier shifted spectrum of $f_o(u)$ is contaminated by undesired multiplicative factors which are contained in the oscillator signal, b(u). The average power/energy spectrum of the product of the envelope waveform, $f_o(u)$, and the random process, b(u), is the Fourier transform of the average auto-correlation function, $R_f(t)$, given in equation (1.4).

---

[1] The Fourier transform of the average auto-correlation function is denoted as the average power spectrum of the average energy spectrum depending on whether the waveform is power limited of energy limited, respectively. Further discussion can be found in [1.20].

The auto-correlation function, $R_f(t)$, which we have described in equation (1.4) can be represented as a product of auto-correlation functions corresponding to other independent random processes [1.21], $b_1(u)$, $b_2(u)$,....., $b_n(u)$; each representing independent phase instability processes. The average auto-correlation function will then be [1.20], [1.21], [1.22]

$$R_{f_n}(t) = R_o(t) \prod_{i=1}^{n} R_{b_i}(t) \qquad (1.5)$$

1.4. Organization of the Dissertation

In chapter two, we consider modeling and characterization of the phase error process and its relationship to the frequency stability measures. Models developed include:

    i) a superposition of a stationary white and a non-stationary random walk phase process and;

    ii) a random walk frequency process along with a random drift phase.

The white phase and random walk phase models will be treated together in one model while the random walk frequency will be treated separately in another model. Thus, the first model treats the effects of white noise on the oscillator while the latter characterizes the oscillator instability caused by the oscillator environment [1.4]. It is shown how moments of the rf signal random process including the auto-correlation function can be evaluated through the use of the phase covariance matrix and the characteristic function.

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In chapter three, conditions are given for stationarity and ergodicity of the mean, auto-correlation function, and power spectral density of the rf signal. Each of the phase instability models mentioned earlier are treated here. Wide sense stationarity is established by providing conditions under which the mean and auto-correlation function of the rf signal are invariant over the time variable, u, for those phase instability models considered. A number of theorems from Papoulis [1.20] pp 328, 330, 343-344, are utilized to determine the ergodicity of our models. It is found that the statistics of the phase appearing the instant at which the observation commences plays an important role in determining the stationarity and ergodicity of the rf signal. For Gaussian distributed instabilities, i.e. white phase, random walk phase, and random walk frequency processes, conditions at the time the rf signal is first observed are determined in establishing ergodicity.

Spectral spreading due to the oscillator instabilities in an envelope modulated signal is discussed in chapter four. Three important modulating waveforms are considered: CW, Infinite Pulse Train, and Finite Pulse Train. Spectral properties are determined by taking the Fourier transform of products of the auto-correlation functions developed in chapter two and time average auto-correlation functions of the particular modulating waveform. In the case of random walk frequency instability, asymptotic expansions are developed to evaluate the related Airy integral which arises in the determination of the spectral spreading.

In chapter five, numerical results and some approximations are presented. An example for each of the major phase models and modulating

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envelopes is worked out and the results are plotted. Finally, our results are summarized along with the significant conclusions in chapter six.

The details of many of the derivations are included in four appendices. In appendix A, it is shown how to determine the n th order moments for the rf signal, $b(u)$. The procedure is based on finding the characteristic function of the vector phase random variable, $\underline{z}$, and setting its argument to the appropriate value.

In Appendix B, we determine the covariance matrix of the phase random process consisting of

    1) the white phase random process and

    2) the random walk phase random process

together as a superposition. The results of Appendix B are used to determine oscillator auto-correlation functions in chapter two.

Appendix C is devoted entirely to finding the covariance matrix of the phase random process resulting from a random walk, i.e. Wiener, frequency process. Due to the mathematical difficulties encountered in undertaking such a task, three approaches were used; the approximating function, the integration formula, and a state space approach. The results of all three approaches were in agreement. Again, these are used in chapter two to determine auto-correlation functions.

In order to prove the theorems stated in chapter three, higher order moments of the rf signal process had to be determined for the phase instability models considered here. This is done in appendix D.

The characteristic function method presented in appendix A provided a powerful tool to determine the moments for the many processes included in the dissertation.

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II MODELING OF PHASE AND FREQUENCY INSTABILITY

2.0. Introduction

As indicated earlier, by frequency instability of a precision frequency source we mean unwanted frequency departures from a nominal value. Frequency stability describes the degree to which a source produces a constant frequency over a specified time interval. In this chapter, we consider the characterization of this time-dependent phenomenon.

A certain class of non-stationary as well as stationary phase random processes will be identified through their probability density functions and their moments, particularly their covariance matrices. This will allow determination of the auto-correlation function of the rf signal, i.e. the oscillator output sinusoidal signal. It will be shown, for such a class of phase random process, that the rf signal is stationary in auto-correlation function despite the basically non-stationary phase driving force. In chapter three, we will establish conditions under which the rf signal is wide sinse stationary.

The method used here provides a different way for determining the time and frequency behavior of oscillators. Most prior work has dealt with the non-stationary aspects of phase through the use of finite time averages (where time shifted auto-correlation functions were not addressed). By commencing with the phase covariance properties, the direct use of non-stationary phase instability leads nonetheless to stationarity of the rf signal auto-correlation function. In a later chapter, the spectral spreading (dispersion) due to phase instability will easily follow from the material presented in this chapter.

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2.1. Oscillator RF Signal

The low pass equivalent, $b(u)$, of the ideal oscillator output waveform, say $r(u)$, is defined through

$$r(u) = Re\{ b(u) \exp(jw_o u) \}$$

where w_o is the oscillator frequency. $b(u)$ may be written

$$b(u) = B e^{j\phi(u)}$$

where B is some arbitrary amplitude and $\phi(u)$ is a deterministic phase function. For a monochromatic oscillator $\phi(u) = \phi$, a constant. It is this low pass equivalent, $b(u)$, which we will consider throughout this dissertation and will refer to it as the "oscillator rf signal" in keeping consistant with the literature [2.1], [2.2]. For non-ideal oscillators found in practice, there will be unwanted departures from the nominal amplitude and frequency. We may express the output oscillator waveform as

$$b(u) = [B + c(u)] e^{j[\phi + z(u)]}$$

where $c(u)$ denotes the amplitude noise and $z(u)$ denotes the phase noise. Both of these are random processes. Since most oscillators are amplitude stabilized [2.3], we shall neglect the amplitude noise and, without loss of generality, assume B to be unity. Also, since ϕ is a deterministic phase, we shall consider the following simplified oscillator signal model.

$$b(u) = e^{j\,z(u)} \qquad\qquad (2.1)$$

In the above oscillator rf signal, the phase random process, $z(u)$, generates the random phase fluctuations. We shall focus our attention on the modeling of this phase error process, $z(u)$. The relationship between the frequency noise and the associated phase random process has been discussed in [2.1]-[2.4]. Basically, a phase random process is the integral of the frequency random process. Translating variances and spectrum from frequency to the phase random process must carefully account for this integral relationship.

Traditionally, the phase error process has been assumed to be a stationary process [2.4],[2.5]. The reason has been the simplicity of such an assumption. However this assumption has limitations as pointed out in [2.4]. Certain idealized models, [2.3], for phase and frequency random processes impose serious limitations on their application to stability analysis by excluding a whole class of phase random processes from such consideration. Examples include the random walk phase and random walk frequency processes.

The Wiener (or BM: Brownian Motion, random walk) phase random process has no derivative and is furthermore non-stationary; in other words it has no instantaneous frequency and cannot be represented by a power spectral density. In this dissertation, these difficulties are overcome by assuming a finite bandwidth oscillator, i.e. a "near white" decorrelation time and by focussing attention on the output signal, $b(u)$.

Since we are dealing with the time and frequency domain properties of the oscillator rf signal itself, we find it appropriate to focus

attention on the sinusoidal output signal, $b(u)$, and its power spectral density, $S(w)$, [2.4]. Knowledge of the power spectral density is of prime importance in applications such as radar Doppler processing, communications, frequency synthesis, and spectroscopy. When the AM noise is negligible and the phase instability is much smaller than one radian, then the low modulation index approximation can be used, i.e. the spectral density of the waveform, $b(u)$, is approximated by the noise sidebands of the phase, $z(u)$. However, the complete formula is often needed to forcast spectral purity when very high ratio frequency multiplication is implemented, [2.4],[2.6].

We also make use of the fact that for a class of non-stationary phase random processes, $z(u)$, the sinusoidal transformation (or operation) yields a random process which is stationary in auto-correlation function. Specifically we obtain stationary auto-correlation properties for the signal, $b(u)$, for the case when first and higher order increments of $z(u)$ are stationary. In chapter III, we determine conditions for a general class of non-stationary phase random processes which lead to wide sense stationary rf signals through the above sinusoidal operation. In a later chapter, we will find that the parameters within the argument of its auto-correlation function have a one-to-one correspondance with the power spectral density. We will make use of this condition.

2.2. Modeling of Phase and Frequency Components

Before we proceed with the modeling we shall describe the properties of the Wiener process[1]. From Ross [2.7] the Wiener process, say $x(s)$, has the following properties:

A stochastic process $[x(s), s \geq 0]$ is said to be a Wiener process with mean drift coefficient, μ, and variance drift coefficient, σ^2, if;

    i. $x(0) = 0$

    ii. $\{x(s), s \geq 0\}$ has stationary independent increments

    iii. For every $s > 0$, $x(s)$ is normally distributed with mean, $(\mu)(s)$, and variance, $\sigma^2 s$.

Since we want the time when the observation is initiated to be arbitrary, our model will be modified as follows:

We define a random walk phase process to contain a phase random variable, $x(u) = X$, corresponding to the condition at the time, $u(-\infty < u < \infty)$, when the observation is initiated. This random variable, X, is assumed to have a Gaussian probability density function where

$$E\{ X \} = \text{an arbitrary constant}$$

and

$$\text{var}\{ X \} = c_1 \sigma_1^2 = \text{an arbitrary constant.}$$

We add to this random variable, an independent Wiener process, say $x_o(t)$, where $x_o(0) \underset{=}{\Delta} 0$, the zero mean drift, $\mu_o = 0$ and the drift variance is σ_1^2. Furthermore, our process is defined to run for negative time as well as positive time, tracing mirror images of paths with equal probability from time, u. Thus, t will be replaced by $|t|$. From, the independence assumption,

$$x(u+t) = X + x_o(t)$$

implies $\qquad var\{x(u+t)\} = var\{X\} + var\{x_o(t)\} = c_1 \sigma_1^2 + \sigma_1^2 |t|$

We shall use the term, $x(u)$, to denote the random walk phase process throughout this dissertation. Next we briefly describe the various types of phase instability models appearing in the literature.

Generally, phase random processes, $z(u)$ (not $b(u)$), are modeled as power law spectral densities. They have been described in the literature and have won wide acceptance [2.4]. The phase instabilities are modeled according to the rate at which the power spectral density of the phase, $z(u)$ (or frequency, $\dot{z}(u)$) random process decays with the transformed variable, f. Hence, if $S_{\dot{z}}(f)$ is the power spectral density of the frequency random process, $\dot{z}(u)$ (the derivative of the phase random process, $z(u)$), then

$$S_{\dot{z}}(f) = \begin{cases} \sum_{a=-2}^{+2} h_a f^a & 0 \le f \le f_h \\ 0 & f > f_h \end{cases}$$

where f_h is an upper cutoff frequency.
The models for various values of $a = -2, -1, 0, 1, 2$ correspond to random

walk frequency, flicker frequency, white frequency/random walk phase, flicker phase, and white phase respectively. The flicker models, i.e. $a = -1, 1$, will not be treated here.

The models we will use tend to have certain physical origins, [2.4], which are as follows:

a) White phase noise, $y(u)$, is usually due to white noise sources external to the *oscillator loop*.

b) Random walk phase (white frequency) noise, $x(u)$, usually arises from additive white noise internal to the oscillator loop.

c) The phase random process, $v(u)$, resulting from random walk frequency noise, $\dot{v}(u)$, is usually related to the oscillator environment such as temperature, vibration and shocks.

Specifically we assume that the phase process is composed of non-stationary components, $v(u) + x(u)$ and stationary components, $y(u)$, so that the phase, $z(u)$, is modeled as

$$z(u) = v(u) + x(u) + y(u) \qquad (2.2)$$

where $\dot{v}(u)$ and $x(u)$ are zero drift mean Wiener processes, characterized as having stationary independent increments and which gives rise to the non-stationary random walk frequency and phase [2.4],[2.5] instability respectively. $y(u)$ is a stationary Gaussian process which gives rise to the white x [2.4],[2.5], instability. Since we are considering finite bandwidth oscillators, the random process, $y(u)$, decorrelates over some interval $-T_0/2 < t < T_0/2$. We will refer to this process as "near

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white". (Note notational change from [2.4],[2.5]; in our work v(u),
x(u) and y(u) are each phase functions).

As indicated earlier, white phase noise, y(u), and random walk phase
noise, x(u), are caused by additive white noise both external and internal
to the oscillator loop where as the random walk frequency, v̇(u), is due
to the oscillator environment. In this dissertation, we shall develop
models for the internal/external white noise induced instability
separately from the instability induced by the oscillator environment.

2.2.1. <u>Model for White Phase and Random Walk Phase</u>

First we consider the models where the phase random process is the
superposition of the stationary white phase process, y(u), and the
random walk phase process, x(u). The means of these phase random
processes can be arbitrary. However, without loss of generality, we

assume that the means are zero$^{\underline{1}}$.

Since oscillators have a finite bandwidth, let us assume a
triangular rather than an impulse auto-correlation function for the
stationary white phase process y(u) as shown in figure 2.1a. This is
consistent with the assumption that y(u) gives rise to the "near white"
instability with the area under the triangular (impulse like) part of
the auto-correlation function being equal to

1.
    When evaluating the auto-correlation function of b(u) it will
become obvious that mean values of x(u), y(u) and v(u), c.f. equation
(2.2), cancel out of the exponential expression. Therefore, the mean
values of the phase random processes do not affect the auto-correlation
function of b(u).
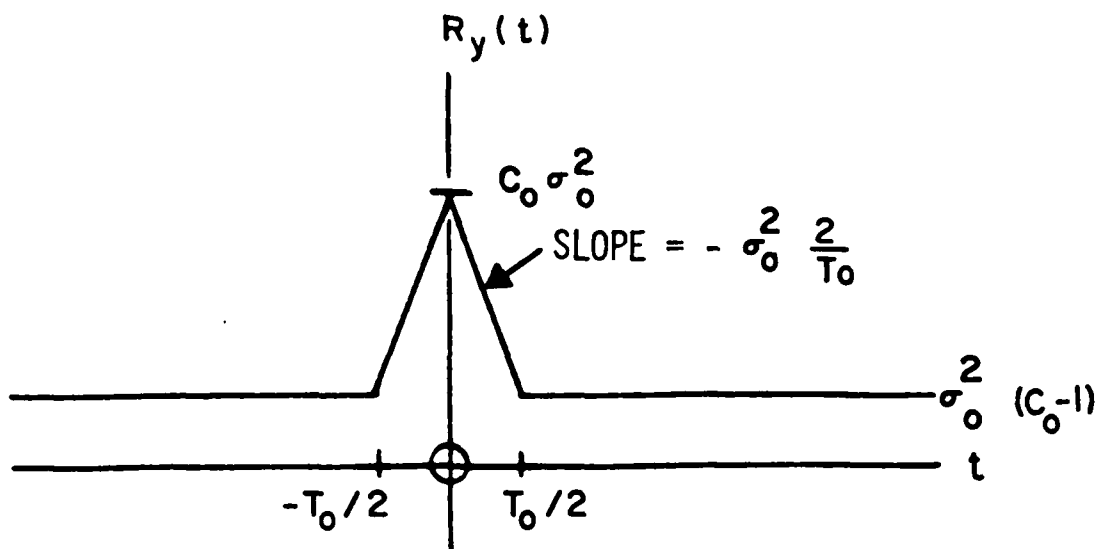
(a)

TRIANGULAR AUTO-CORRELATION FUNCTION
STATIONARY NEAR WHITE PHASE STABILITY
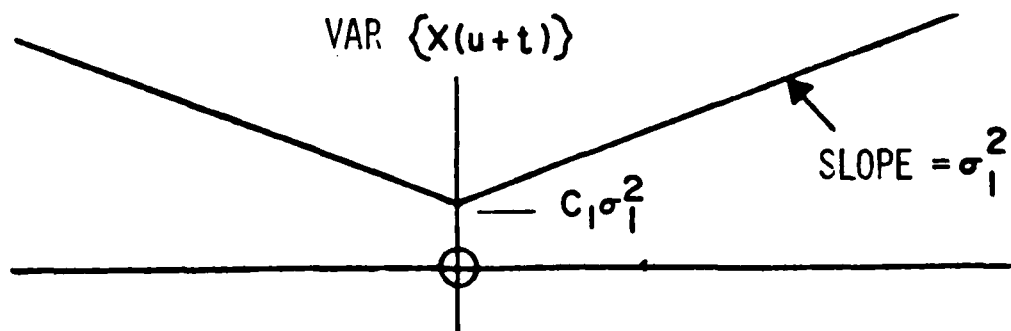
RANDOM WALK COMPONENT CORRELATION
OF STABILITY FACTOR $X(u)$
(b)

FIGURE 2.1. STATIONARY AND NON-STATIONARY STABILITY

$$\sigma_o^2 .$$

As seen in figure 2.1a, we have:

$$\text{Var}\{y(u)\} = \text{Var}\{y(u+t)\} = c_o \sigma_o^2 \qquad (2.3a)$$

$$\text{Cov}\{y(u+t) \; y(u)\} = E\{y(u+t) \; y(u)\} \qquad (2.3b)$$

$$\text{Cov}\{y(u+t) \; y(u)\} = \sigma_o^2 (c_o - \frac{2}{T_o}|t|) \qquad : \; |t| < \frac{T_o}{2}$$

$$= \sigma_o^2 (c_o - 1) \qquad : \; |t| > \frac{T_o}{2} \qquad (2.3c)$$

where c_o is arbitrary[1].

---

[1] The phase random processes, $x(u)$, $y(u)$ and $v(u)$ are functions of time, u. However, it will become obvious that in the case of the rf signal, $b(u)$, its auto-correlation function depends only on the phase change from time, u, to time, u+t, and not on the value or statistics at time, u, $-\infty < u < \infty$. We will therefore associate arbitrary values, c_o, c_1 and c_3 to the phase statistics at time, u.

The variance of the random walk component is shown in figure 2.1b. From the definition of our random walk process, with u being the reference time, the following can be written [2.8] - [2.11].

$$\text{Var}\{x(u)\} = c_1 \sigma_1^2 \qquad (2.4a)$$

$$\text{Var}\{x(u+t)\} = (c_1 + |t|) \sigma_1^2 \qquad (2.4b)$$

$$\text{Cov}\{x(u+t)\, x(u)\} = c_1 \sigma_1^2 \qquad (2.4c)$$

where σ_1^2 [1] is the drift variance of the random walk (Wiener) component and c_1 is arbitrary. This assumes that the variance of $x(u)$ at some reference point, u, is arbitrary and that stationary zero mean independent Gaussian increments (whose variance is proportional to t) are additive to the process with increasing or decreasing t. The independent increment property can be used to derive equation (2.4c) as follows [2.7];

$$\text{Cov}\{x(u+t)\, x(u)\} = E\{[x(u+t) - x(u) + x(u)]\, [x(u)]\}$$

$$= E\{[x(u+t) - x(u)]\, [x(u)]\} + E\{[x(u)]\, [x(u)]\}$$

$$= 0 + c_1 \sigma_1^2 = c_1 \sigma_1^2$$

which is just the variance of the random variable, $x(u)$, defined at the beginning of this section at time = u, i.e. t = 0. Later we shall see

---

that the arbitrary variance $c_1 \sigma_1^2$ and $c_o \sigma_o^2$ which we assign to $x(u)$ and $y(u)$ respectively will not affect the auto-correlation function of the oscillator rf signal, $b(u)$. When determining the auto-correlation and spectral properties for the case when the envelope waveform, $f_o(u)$, is constant, i.e. a cw, the oscillator output itself, we use the complete description of equations (2.3) and (2.4). For the case when $f_o(u)$ is a train of pulses we can assume that the phase, $z(u)$, is constant within each of the small intervals of $d(u-iT)$ denoted by $\triangle u$. In doing this, we are assuming that the pulse width, $\triangle u$, is much smaller than the interpulse period, T, i.e. $\triangle u \ll T$.

There are two major approaches for the frequency stability characterization. One deals with the characterization in the time domain and the other with the characterization in the frequency domain. Translations between the two domains have also been described in the literature [2.4],[2.5]. For the most part, this has been limited to time averages of the phase/frequency random processes including different ways of determining their variances. Equivalent spectral domain representations of the phase/frequency random processes include the use of transformed window functions acting on the phase process itself resulting in sine-x-over-x convolutions in the spectral domain. In this chapter, we shall focus on the details of the time domain rf signal auto-correlation function. This will set the stage for spectral domain characterization. The relationship between the two domains will thus be established in the subsequent chapters. The power spectral density of the complete oscillator signal output, $b(u)$, as affected by the phase random process, $z(u)$, will be determined. Characterization in the frequency domain is important both for theoretical reasons and for

application purposes. Spectral spreading is a primary specification for engineers in spectroscopy and radar and it is therefore a widely accepted chatacterization of oscillator stability.

The oscillator waveform was written in equation (2.1) as a low pass equivalent to a sinusoidal random process containing the phase process, z(u), as follows

$$b(u) = e^{j\,z(u)} \qquad\qquad (2.1)$$

The modulating envelope waveform, $f_o(u)$, may be written

$$f_o(u) \triangleq c \qquad\qquad (2.5a)$$

for the cw case and

$$f_o(u) \triangleq c \sqrt{\frac{T}{\Delta u} \sum_{i=-N/2}^{N/2-1} d\left(u - [\,i + \frac{1}{2}\,]\,T\right)} \qquad\qquad (2.5b)$$

for the pulse train of N (even) pulses where the interval, $\triangle u$, and the function, d(u), are defined through

$$\lim_{\triangle u \to 0} \frac{1}{\triangle u} \int_{-\infty}^{\infty} d(u)\,du = 1$$

and

$$d(u) = 0 \qquad : \quad |u| > \frac{\triangle u}{2}$$

From equations (1.2), (1.3) and (1.4) we may evaluate the average auto-correlation function, $R_f(t)$, of the system output waveform, f(u), by forming the product of the auto-correlation function of the oscillator output waveform, equation (2.1), and the time average

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auto-correlation function of the modulating envelope waveform, equation (2.5). The auto-correlation function of the oscillator output waveform, equation (2.1), is [2.12],[2.13],[2.14]

$$R_b(t) \triangleq E\{b(u + t) b^*(u)\}$$

$$= E\{e^{j[z(u+t)]} e^{-j[z(u)]}\}$$

$$= E\{e^{j[z(u+t) - z(u)]}\}$$

$$= E\{e^{j[x(u+t) + y(u+t) - x(u) - y(u)]}\}$$

The argument of the exponential can be expressed in vector notation by defining

$$\underline{z} = \underline{x} + \underline{y} \triangleq \begin{bmatrix} x(u+t) \\ x(u) \end{bmatrix} + \begin{bmatrix} y(u+t) \\ y(u) \end{bmatrix} \tag{2.6}$$

and

$$\underline{p} \triangleq j [p_1, \ p_2]^T$$

In Appendix A, we illustrate the use of characteristic functions for the evaluation of higher order moments of the rf signal, b(u). Thus the auto-correlation function of the oscillator rf signal is written using the procedure outlined in Appendix A. We have

$$R_b(t) \triangleq E \{ b(u+t)b^*(u)\} = E (e^{jz(u+t)-jz(u)})$$

which in vector form is

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$$R_b(t) = E\{ e^{\underline{z}^T \underline{p}} \} \triangleq F_{\underline{z}}(\underline{p}) = \exp\{-\frac{1}{2}(m_{11} + m_{22} - 2m_{12})\} \qquad (2.7)$$

which is the characteristic function, F, of $\underline{z}$ evaluated at

$$\underline{p} = j\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

and where m_{ij} is the ij th element of the covariance matrix, $M_{\underline{z}}$.

In vector notation for $z(u)$, the joint density function of a Gaussian random process, $\underline{z}$, is [2.7], [2.14]

$$p_{\underline{z}}\{\underline{z}\} = \frac{1}{2\pi |M_{\underline{z}}|^{1/2}} \exp\{-\frac{1}{2} \underline{z}^T M_{\underline{z}}^{-1} \underline{z}\} \qquad (2.8)$$

where the superscript, T, denotes transpose. The above covariance matrix, $M_{\underline{z}}$, which is derived from equation (2.3) and equation (2.4) through Appendix B is given by

$$M_{\underline{z}} = M_{\underline{x}} + M_{\underline{y}} = \begin{bmatrix} \sigma_1^2(c_1 + |t|) + \sigma_o^2 c_o & \sigma_1^2 c_1 + \sigma_o^2(c_o - \frac{2}{T_o}|t|) \\ \sigma_1^2 c_1 + \sigma_o^2(c_o - \frac{2}{T_o}|t|) & \sigma_1^2 c_1 + \sigma_o^2 c_o \end{bmatrix} \qquad (2.9a)$$

for $|t| < \frac{T_o}{2}$ and

$$M_{\underline{z}} = \begin{bmatrix} \sigma_1^2(c + |t|) + \sigma_o^2 c & \sigma_1^2 c + \sigma_o^2(c - 1) \\[2ex] \sigma_1^2 c + \sigma_o^2(c - 1) & \sigma_1^2 c + \sigma_o^2 c \end{bmatrix} \qquad (2.9b)$$

for $|t| > \dfrac{To}{2}$

We now find the auto-correlation function of the oscillator rf signal, $R_b(t)$, using equation (2.7) and the value of $\underline{p}$ given above. We obtain:

$$R_b(t) = E\{e^{j[z(u+t)-z(u)]}\} = E\{e^{\underline{z}^T \underline{p}}\}$$

$$= \int_{-\infty}^{\infty} \exp\{\underline{z}^T \underline{p}\}\, p_{\underline{z}}\{\underline{z}\}\, d\underline{z}$$

which gives

$$R_b(t) = \exp\{-\frac{1}{2}(\sigma_1^2 |t| + 2\sigma_o^2 \frac{2}{To}|t|)\} \quad : |t| < \frac{To}{2}$$

$$= \exp\{-\frac{1}{2}(\sigma_1^2 |t| + 2\sigma_o^2)\} \qquad\qquad : |t| > \frac{To}{2} \quad (2.10)$$

The logarithm of $R_b(t)$ is shown in figure 2.2.

$$\ln\{R_b(t)\}$$

t

$T_0/2$

$\text{Slope} = -\left(\dfrac{\sigma_1^2}{2} + \dfrac{\sigma_0^2 \, 2}{T_0}\right)$

$-\left(\dfrac{\sigma_1^2 \, T_0}{4} + \sigma_0^2\right)$

$\text{Slope} = -\dfrac{\sigma_1^2}{2}$
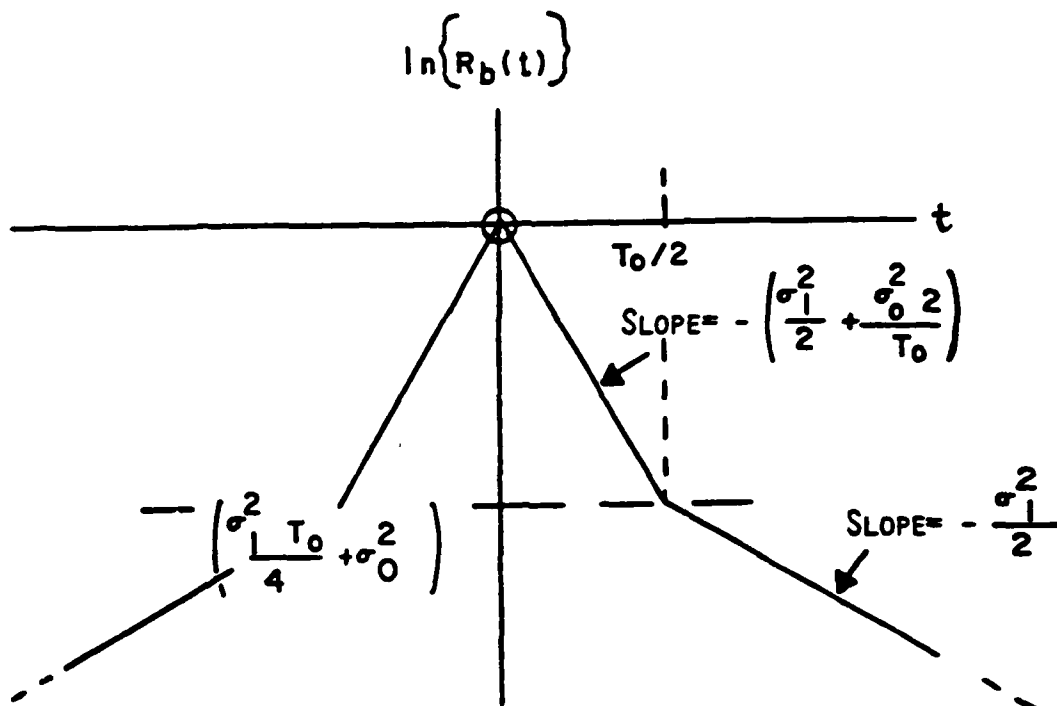
FIGURE 2.2 WHITE PHASE AND RANDOM WALK PHASE, LOGARITHM OF
AUTO-CORRELATION, R (t) OF OSCILLATOR WAVEFORM b(u)
            b

It is well known [2.14], [2.15] that the Fourier Transform of the auto-correlation function gives the power spectral density when the process is a stationary random process. We establish conditions for stationarity and ergodicity in chapter III and proceed to evaluate power spectral densities as well as average power/energy spectrums for different modulating envelopes in chapter IV.

2.2.2. Model for Random Walk Frequency

As mentioned previously random walk frequency is yet another instability factor identified in oscillators. This is caused by the environment in which the oscillator must operate. The random walk frequency model is characterized by

$$\text{var}\{\dot{v}(u)\} = c_2 \sigma_2^2 \tag{2.11a}$$

$$\text{var}\{\dot{v}(u+t)\} = (c_2 + |t|)\sigma_2^2 \tag{2.11b}$$

$$\text{cov}\{\dot{v}(u+t)\,\dot{v}(u)\} = c_2 \sigma_2^2 \tag{2.11c}$$

where $c_2 \sigma_2^2$ is the variance of the frequency offset[1] due to the effects of long term drift over time, u. The frequency variance term due to the drift variance coefficient has a linear slope as shown in figure 2.3a. The variance of the resulting phase has a third power relationship with time as shown in figure 2.3b and proved in Appendix C. The properties of the rf signal associated with this process can be investigaced in much the same manner as in the case of the above random walk phase (white frequency) component. We will first determine the covariance matrix of the phase random process associated with the random walk frequency.

---

[1] Unlike the arbitrary constants, c_o, c_1 and c_3 (associated with the phase statistics for any u; $-\infty < u < \infty$) which have no effect on the auto-correlation function of the rf signal, the constant, c_2, (associated with the long term frequency linear drift over all u; $-\infty < u < \infty$) does affect the auto-correlation function over the short term time interval, denoted by t. It will be shown later that this term can be treated as a constant over time, t, if the change with respect to time, u, is very slow.

VAR ($\dot{V}[u+|t|]$)

u
t ≠ 0
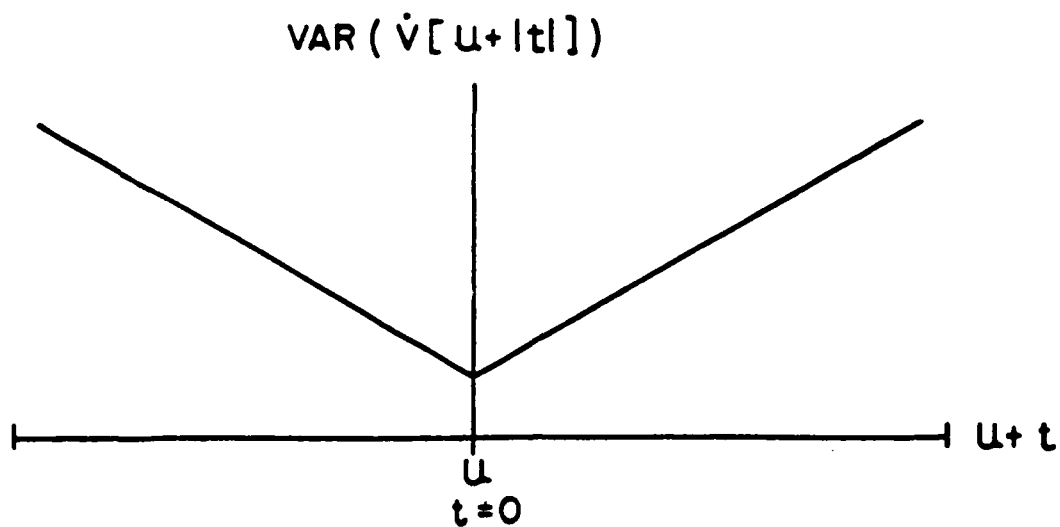
(a) VARIANCE OF RANDOM WALK FREQUENCY

VAR ($V[u+|t|]$)

u+ t

u
t = 0
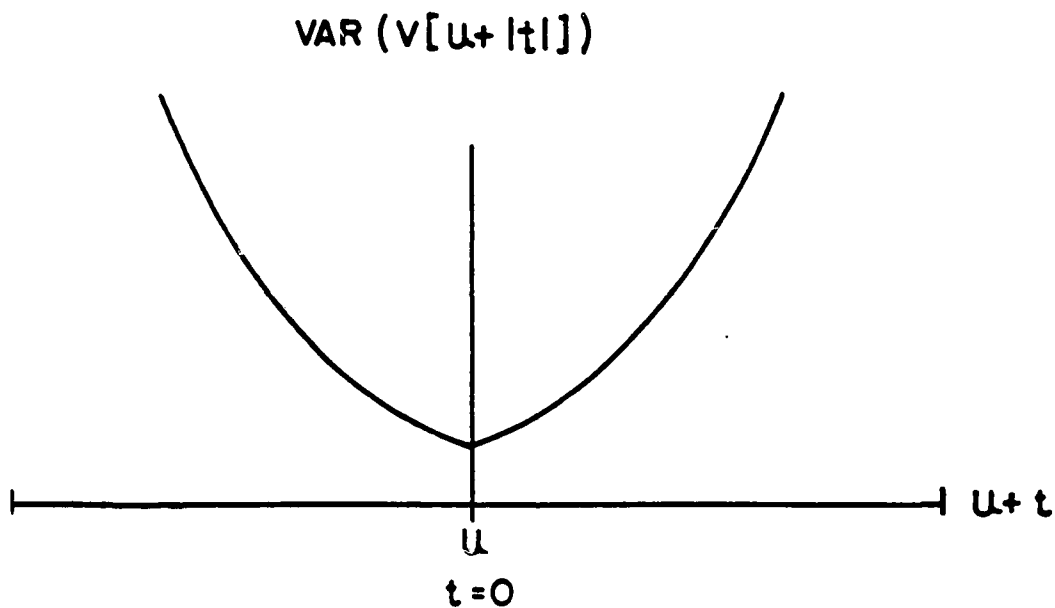
(b) VARIANCE OF RESULTING PHASE PROCESS

FIGURE 2.3 FREQUENCY RANDOM WALK PROPERTIES

Using this covariance matrix, the auto-correlation function of the oscillator signal containing the random walk frequency factor will be derived next through the characteristic function approach given in Appendix A. Third, the power spectral density is obtained in chapter IV by taking the Fourier transform of the auto-correlation function. Spectral spreading due to the random walk frequency component can then be determined.

In much the same fashion as in Section 2.2.1, we find the auto-covariance matrix of the process, $v(u)$, which results from the random walk frequency, $\dot{v}(u)$. The phase random process may consist of an arbitrary phase random variable, $v(u_o)$, including accumulated drift effects [2.1],[2.2] up to some time, u_o, plus the integral of the frequency random process, $\dot{v}(u)$.

$$v(u) = v(u_o) + \int_{u_o}^{u_o+u} \dot{v}(h)\, dh \qquad (2.12)$$

This is a non-stationary random process for which the auto-covariance matrix can be derived as

$$M_{\underline{v}} \triangleq E\left\{\underline{v}\,\underline{v}^{*T}\right\} \triangleq E\left\{\begin{bmatrix} v(u+t) \\ v(u) \end{bmatrix}\begin{bmatrix} v^*(u+t) & v^*(u) \end{bmatrix}\right\} \qquad (2.13)$$

From (2.13) we see that we must find

$$\text{var}\{v(u)\} = E\{v(u)\,v^*(u)\}$$

$$\text{var}\{v(u+t)\} = E\{v(u+t)\,v^*(u+t)\}$$

$$\text{cov}\{v(u+t)\,v^*(u)\} = E\{v(u+t)\,v^*(u)\}$$

The above expectations have been evaluated in Appendix C whereby the covariance matrix for the phase random processes, $v(u)$, is given by

$$M_{\underline{v}} = \frac{\sigma_2^2}{2}
\begin{bmatrix}
c_3 + c_2|t| + \dfrac{2}{3}\dfrac{|t|^3}{3} & c_3 \\[2ex]
c_3 & c_3
\end{bmatrix}
\tag{2.14}$$

where σ_2^2 is the random walk frequency drift variance, $c_2\,\sigma_2^2$ is the variance of the frequency due to long term drift, and $c_3\,\sigma_2^2$ is arbitrary.

Since the process, $v(u)$, is Gaussian, we can use Appendix A, equation (A.3b), to evaluate the auto-correlation function, $R_b(t)$, of an oscillator RF signal, $b(u)$. This is a random process, whose auto-correlation function is invariant with time, u, and dependent only on the shift variable, t. This property of $R_b(t)$, based on the covariance matrix of equation (2.14), will be established in greater detail through Theorem I and Theorem II in chapter III.

$$R_b(t) = \exp\left\{ -\frac{\sigma_2^2\left(3c_2|t|^2 + |t|^3\right)}{6} \right\}
\tag{2.15}$$

2.2.3. Phase and Frequency Linear Drift

Frequency drift due to aging of the oscillator results in an extremely slow variation of the frequency process with respect to time. This can be approximated by a linear term about some local point [2.1], [2.2]. Let us define the total phase random process as follows:

$$\dot{z}_T(u) = a + bu + \dot{z}(u) \tag{2.16}$$

We can proceed to find the auto-correlation function of the rf signal. Assuming a and b are random variables and $\dot{z}(u)$ is an independent, zero mean, random process, all Gaussian, where z(u) contains the processes as described previously in equation (2.2), we determine the phase random process of the entire signal.

$$z_T(u) = c + au + \frac{b}{2}u^2 + z(u) \tag{2.17}$$

The covariance matrix is defined by

$$M_{\underline{z}_T} \triangleq E(\underline{z}_T \underline{z}_T^T) = E\left\{ \begin{bmatrix} z_T(u+t) \\ z_T(u) \end{bmatrix} \begin{bmatrix} z_T(u+t) & z_T(u) \end{bmatrix} \right\}$$

Now $\sigma_c^2 \triangleq E(c^2)$, $\sigma_a^2 \triangleq E(a^2)$, $\sigma_b^2 \triangleq E(b^2)$.

Also, time, u and t, are non-random, i.e.

$$E(u^k t^\ell) = u^k t^\ell : \quad k, \ell = 0, 1, 2, \ldots\ldots$$

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Furthermore, the covariance matrix of $\underline{z}$ has been given for each of its
independent components, $\underline{x}$, $\underline{y}$, and $\underline{z}$, through equation (2.9) and equation
(2.14). Then the covariance matrix of $\underline{z}_T$, above is easily written:

$$
M_{\underline{z}_T} =
\begin{bmatrix}
\sigma_c^2 + \sigma_a^2(u+t)^2 + \dfrac{\sigma_b^2}{4}(u+t)^4 + m_{11} & \sigma_c^2 + \sigma_a^2 u(u+t) + \dfrac{\sigma_b^2}{4} u^2(u+t)^2 + m_{12} \\[4mm]
\sigma_c^2 + \sigma_a^2 u(u+t) + \dfrac{\sigma_b^2}{4} u^2(u+t)^2 + m_{21} & \sigma_c^2 + \sigma_a^2 u^2 + \dfrac{\sigma_b^2}{4} u^4 + m_{22}
\end{bmatrix}
$$

(2.18)

where m_{ij} is the ij th element of the covariance matrix, $M_{\underline{z}}$, defined
in Appendix A, and c is an independent Gaussian random variable. The
auto-correlation function of the rf signal is found by taking the
characteristic function

$$
R_b(u,t) = F_{\underline{z}_T}(\underline{p}) = \exp\left\{ -\frac{1}{2}\, \underline{p}^T\, M_{\underline{z}_T}\, \underline{p} \right\}
$$

$$
= \exp\left\{ -\frac{1}{2}\left(\sigma_a^2 t^2 + \sigma_b^2 t^2 [2u+t] + \sum m_{ij} \right)\right\} \qquad (2.19)
$$

where $\underline{p}^T = [\ +1,\ -1\ ]$ and the appropriate m_{ij} are obtained from
equation (2.18) through the formula in equation (2.7).
The variance of the phase accumulated up to time u is σ_c^2 and the
variance of the frequency accumulated up to time u is $\sigma_a^2 + \sigma_b^2 u^2$.

A further consideration of equation (2.19) suggests that the first and

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second order drift terms (first and second terms in the exponent of equation (2.19)) can be approximated as follows:

$$(\sigma_a^2 + \sigma_b^2[2u+t]^2)t \simeq (\sigma_a^2 + [2u\sigma_b]^2)t \simeq \sigma_{drift}^2(u,t)\,t \qquad (2.20)$$

for $0 < t \ll u$. The drift variance resulting from this approximation is lumped into $c\,\sigma^2$ for the offset frequency in the treatment of the random walk frequency instability model in Section 2.2.2. In general, it is a function of u and t where u gives rise to long term drift and t is the short term shift variable which has a negligible effect on the drift variance. From equation (2.16), it has an assumed Gaussian distribution.

It should be noted that the expressions in the exponent of the auto-correlation functions, equations (2.10), (2.15), and (2.19) represent the first increment structure function with $t1 = t2 = t$. Structure functions used by Lindsey and Lewis [2.1] were first introduced by Kolmogorov [2.16]. The first increment structure function of a phase random process (instability) is defined by

$$D_z(u\,;t_1,t_2) \triangleq E\{[z(t_1 + u) - z(u)][z(t_2 + u) - z(u)]\}$$

The structure functions are useful in obtaining phase stability measures [2.1] as we shall demonstrate in the following section.

2.3. Frequency Stability Measures

Thus far in this chapter, we have considered a non-stationary model for the phase random process from which the covariance matrices and auto-correlation functions of the rf signal followed. Now we consider

phase stability measures to describe precision frequency sources and how their frequency instability affects system performance.

An appropriate measure to use in conjunction with phase instability models discussed in this chapter is based on the work of Lindsey and Lewis [2.1] and Lindsey, et al [2.2]. This involves the elements of the phase covariance matrix developed throughout this dissertation. The measures are dimensioned in phase standard deviation and they provide a quantitative description of the phase deviation with respect to time. For all the models discussed in this chapter, the covariance matrices have been derived and the frequency stability measure can be computed in a straight forward manner through use of the structure function. From [2.1], [2.2] we have

$$dz(t) \triangleq \sqrt{D_z(t)} \triangleq \sqrt{D_z(u \; ; \; t+u \; , \; t+u \;)} \qquad (2.21a)$$

Generally, in terms of the covariance matrix elements, m_{ij}, of M_z, given in equations (2.9), (2.14) and (2.18), we have

$$dz(t) = \sqrt{m_{11} + m_{22} - 2m_{12}} = \sqrt{-2 \ln R_b(t)} = \sqrt{-2 \ln F_z(1, -1)} \qquad (2.21b)$$

This is the square root of the exponent in the auto-correlation function; given that the reference time for all processes is u. In Table 2.1, we provide the square of the frequency stability measures of equation (2.21) for each of the processes considered in this chapter.

Next, we investigate stochastic properties of the rf signal, b(u), as affected by our phase instability models. Stationarity and ergodicity in the mean, auto-correlation function and power spectral density are all considered.

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TABLE 2.1 MEASURES OF PHASE INSTABILITY

PROCESS	$-2 \ln F_{\underline{z}}(\underline{p}) = D_{\underline{z}}(t)$
WHITE PHASE	$\dfrac{4}{T_0} \sigma_0^2 \, \lvert t \rvert \quad : \lvert t \rvert < \dfrac{T_0}{2}$ $2\sigma_0^2 \qquad : \lvert t \rvert > \dfrac{T_0}{2}$
STARTING PHASE	0
RANDOM WALK PHASE	$\sigma_1^2 \, \lvert t \rvert$
STARTING FREQUENCY	$\sigma_a^2 \, \lvert t \rvert^2$
RANDOM WALK FREQUENCY	$\dfrac{1}{3} \sigma_2^2 \, \lvert t \rvert^3$
LINEAR DRIFT FREQUENCY	$(2 u \sigma_b)^2 \, \lvert t \rvert^2$

NOTE:

$u \triangleq$ LONG TERM

$t \triangleq$ SHORT TERM

$$\underline{p}^T = \begin{bmatrix} j, & -j \end{bmatrix}$$

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III. STATIONARITY AND ERGODICITY OF THE OSCILLATOR RF SIGNAL

3.0. Introduction

In the previous chapter, we discussed three models for the phase random process, i.e. "near white" phase, random walk phase, and random walk frequency random processes. In this chapter, we will study the stationarity and ergodicity properties [3.1], [3.2] of the rf signal driven by the above three phase random processes. Conditions will be established for the stationarity and ergodicity in the mean, auto-correlation function, and power spectral density. These results will be useful in the justification of what is to follow in chapter four, i.e., the determination of the power spectral density through the Fourier transform of the auto-correlation functions. First, we consider the stationarity properties.

3.1. Stationarity

We now examine conditions for stationarity of the rf signal. Rutman [3.3], and Boileau and Picinbono [3.4] and others have called attention to the fact that one should not misuse the stationarity assumption or the correlation function, $R(t)$, that has been widely used in the frequency stability literature. Although it is convenient to assume stationarity, one must check that this is not in conflict either with other parts of the model or with some physical arguments [3.3]. As already pointed out, our models for the phase random process, $z(u)$, are generally non-stationary. Since our attention is being focused primarily on the oscillator rf signal,

$$b(u) \triangleq e^{j \, z(u)} \tag{3.1}$$

we will proceed to establish the stationarity of $b(u)$.

Definition 1:

A random process which satisfies

$$E\{b(u)\} = \eta \tag{3.2a}$$

and
$$R_b(t) \triangleq E\{\, b(u+t) \, b^*(u) \, \} \tag{3.2b}$$

for all u is said to be stationary in the wide sense [3.5].

We now establish the following definition and theorem for the random processes $z(u)$ and $b(u)$.

Definition 2:

The vector random process, $\underline{z}(u,t)$, is a two element vector of the random process, $z(u)$, taken at time $(u+t)$ and u; i.e.

$$\underline{z}(u,t) \triangleq \begin{bmatrix} z(u+t) \\ z(u) \end{bmatrix} \tag{3.3}$$

3.1.1. Invariance in the Mean with Time

Let $z(u)$ be a zero mean (stationary or non-stationary) Gaussian random process, written $z(u) = z_o + z_1(u)$, where z_o, is a random variable which we shall call a reference phase and $z_1(u)$ contains the white and independent increment (random walk) models, independent from z_o.

In the treatment of each of the phase random processes throughout all other chapters of this dissertation, second order moments of the rf

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signal, $b(u)$, are used. This requires two time variables denoted by u and $u+t$. We also assume arbitrary phase conditions at reference time, u, whose variances are associated with the multipliers, c_o, c_1, and c_3, for the "near white" phase, random walk phase, and random walk frequency random processes respectively. It is shown that these terms cancel out in the expressions for the auto-correlation function of $b(u)$.

Since, in this section, we are considering only the first order moment, we need only one time variable denoted, u, with reference time, 0. For convenience, we include all the terms constituting the phase conditions at the reference time into the term, z_o, and assign it an arbitrary variance multiplier, c. The term, $z_1(u)$, is due to the phase generated by the independent white and increment processes occurring from time, 0, to time, u. Our conditions are;

$z_1(0) = y(0) - z_o$, the value of the "near white" phase process, $y(u)$, at time, 0, less the arbitrary phase component, z_o, whose variance is

$$\text{var}(z_o) = c \, \sigma_{zo}^2 \triangleq c_1 \sigma_1^2 + (c_o - 1)\sigma_o^2 + c_3 \sigma_2^2.$$

Also

$$\text{var}(z_1(u)) = \sigma_1^2 |u| + \sigma_o^2 + c_2 \sigma_2^2 |u| + \sigma_3^2 \frac{|u|^3}{} \triangleq \sigma_{z1}^2 (u).$$

We see that

$$\text{var}(z_1(u)) = \sigma_{z1}^2 (u) < \infty, \text{ for any } |u| < \infty.$$

Theorem 1: If $\text{var}(z_o) \longrightarrow \infty$, then the <u>mean of the rf signal</u>, $b(u)$, is <u>invariant with time</u>, i.e.

$$E(b(u)) \triangleq E\{ e^{j(z(u))} \} = \mathcal{R} \qquad (3.4)$$

<u>Proof:</u> The mean, $E(b(u))$ is

$$E(b(u)) = \frac{1}{\sqrt{2\,\Pi}\,\sigma_z(u)} \int_{-\infty}^{\infty} \exp\left(jz - \frac{z^2}{2\sigma_z^2(u)} \right) \, dz = E(\exp\{jz\})$$

where $\sigma_z^2(u) \triangleq c\,\sigma_{zo}^2 + \sigma_{z1}^2(u)$.

In the characteristic function, $F_z(p)$, if we set $p = 1$, we obtain

$$F_z(1) \triangleq E(\exp\{ jz(u)\}) = \exp\{ -\frac{1}{2} m_{11} \}$$

$$E(b(u)) = \exp\{-\frac{1}{2}\sigma_z^2(u)\} = \exp\{-\frac{1}{2}[c\,\sigma_{zo}^2 + \sigma_{z1}^2(u)]\} \qquad (3.5a)$$

Taking the following limit we have the relationship

$$\lim_{c\,\sigma_{zo}^2 \longrightarrow \infty} E(b(u)) = 0 \qquad (3.5b)$$

which shows that $E(b(u))$ is invariant with u. Q.E.D.

We know that the oscillator phase is a renewal process and takes on values in the range $[-\Pi, \Pi]$. In our discussion thus far, we have assumed Gaussian phase processes where the variable takes on values in the range $(-\infty, \infty)$. We should, therefore, restrict the phase variable by the modulo 2Π operation to the range $[-\Pi, \Pi]$. We now state the following lemma which will be obtained from Theorem 1.

Lemma 1: The average value

$$E(b(u)) = E(\exp[jz(u)])$$

is invariant with time if the probability density function of z_o is uniformly distributed with

$$p(z_o) = \frac{1}{2\Pi} \qquad\qquad : \quad |z_o| < \Pi \qquad\qquad (3.6)$$

Proof: From theorem 1, one needs only to convert the variable of the Gaussian density function to modulo 2Π.

For a particular point, say x within $-\Pi < x < \Pi$, the modulo 2Π value is the superposition of all $z_o = x + n\, 2\Pi$; where n is

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$n = -\infty, \ldots -2, -1, 0, 1, \ldots \infty$, as shown in figure 3.1. In terms of the original Gaussian function,

$$p(z_o) = \frac{1}{\sqrt{2\pi c}\ \sigma_{zo}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(x - n2\pi)^2}{2 c\ \sigma_{zo}} \right]$$

Multiplying and dividing through by $\sigma_{zo} / 2\pi$ we have

$$p(z_o) = \frac{1}{2\pi}\ \frac{\sigma_{zo}}{\sqrt{2\pi}\ \sigma_{zo}} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{1}{2c}\left[\frac{2\pi n}{\sigma_{zo}} - \frac{x}{\sigma_{zo}}\right]^2\right\} \frac{2\pi}{\sqrt{c\sigma_{zo}}}$$

Then from the approximating function[1] and setting

$$\frac{2\pi}{\sqrt{c}\ \sigma_{zo}} = \Delta u \quad \text{and} \quad \frac{2\pi}{\sqrt{c}\ \sigma_{zo}}\ n = u, \quad \text{we have for } |z_o| < \pi.$$

$$\lim_{\substack{c\sigma_{zo} \to \infty \\ \Delta u \to 0}} p(z_o) = \frac{1}{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left[u - \frac{x}{\sqrt{c\sigma_{zo}}}\right]^2\right] du\right] = \frac{1}{2\pi} \quad (3.7)$$

$$\text{Q.E.D.}$$

---

[1] The approximating function approach allows integration by the summation of discrete values in the limit. The details of the approach can be found in [3.6].
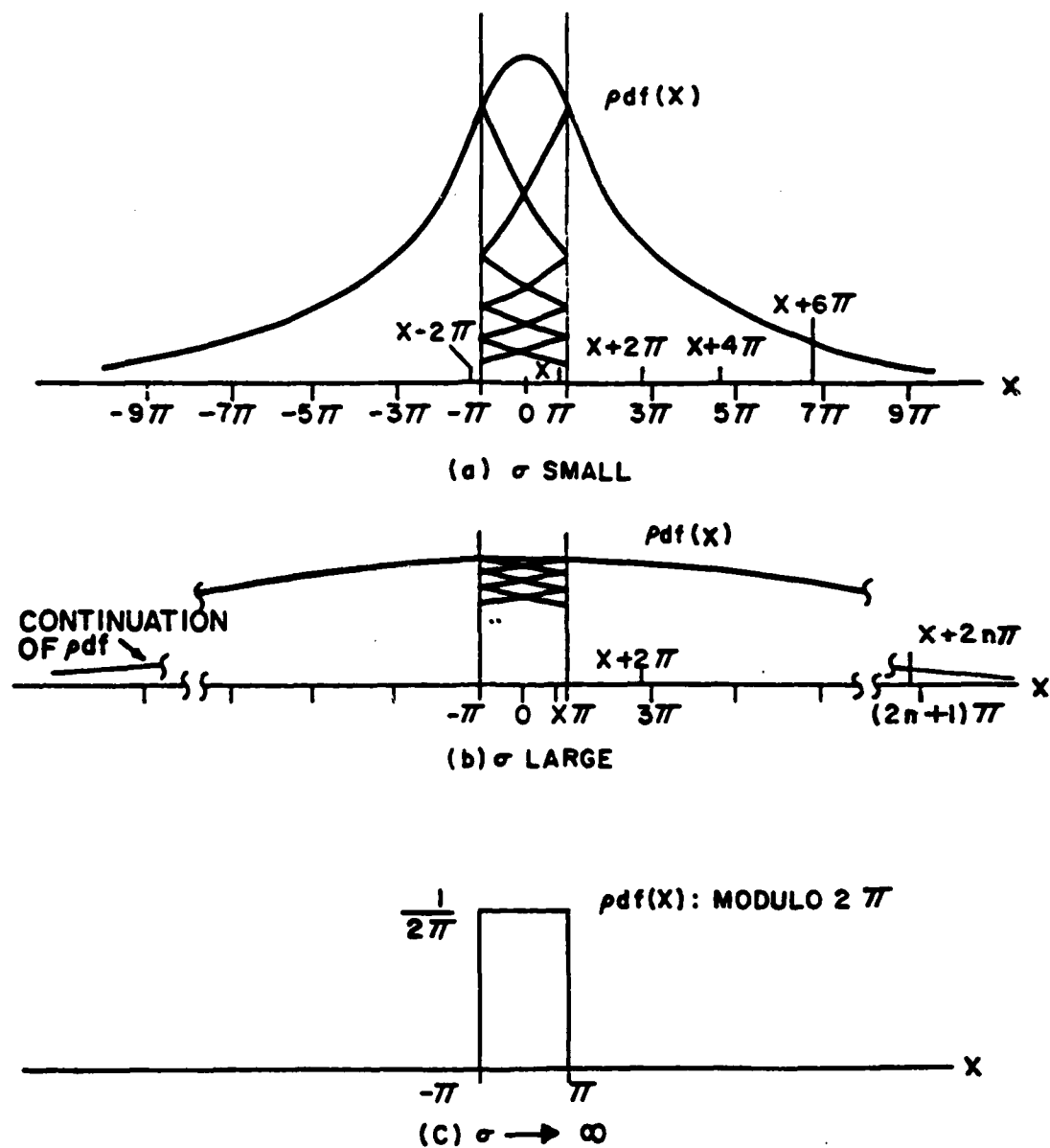
FIGURE 3.1   SUPERPOSITION OF pdf SEGMENTS OF 2 π FROM
GAUSSIAN DENSITY FUNCTION

It should be noted that an oscillator which has been drifting over
a long period of time will have a reference phase which is completely
uncertain, i.e. it has a uniform distribution over 2 π radians. This
should be an obvious condition that one would expect.

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We can also find the average value of $b(u)$ by using the density function of equation (3.6) as follows

$$E(b(u)) = \int_{-\pi}^{\pi} \exp\{jz(u)\} \frac{1}{2\pi} dz(u) = \frac{1}{j2\pi} \Big|_{-\pi}^{\pi} \exp\{jz\}$$

$$= \frac{\exp\{j\pi\} - \exp\{-j\pi\}}{j2\pi} = \frac{2\sin(\pi)}{2\pi} = 0 \qquad (3.8)$$

which agrees with the result from equation (3.5).

3.1.2. Invariance in Auto-correlation Function with Time, u

Let the vector random process $\underline{z}(u,t) = \underline{z}$ of real elements have a covariance matrix of the form

$$E(\underline{z}\,\underline{z}^T) = \begin{bmatrix} a(u) + g(t) & a(u) - h(t) \\ a(u) - h(t) & a(u) \end{bmatrix} \qquad (3.9)$$

The rf oscillator signal, $b(u)$, is given by

$$b(u) = \exp\{ j\, z(u) \} \qquad (3.10)$$

where $z(u)$ is a Gaussian phase random process which is not necessarily stationary and u denotes real time. The auto-correlation function is given by

$$R_b(t) = E(\exp\{ j[z(u+t) - z(u)]\}) \qquad (3.11)$$

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<u>Theorem 2</u>: The auto-correlation function of the rf oscillator signal,

b(u), with the covariance matrix given by equation (3.9) depends

only on t for all u and is given by

$$R_b(t) = E(b(u+t) b^*(u)) = \exp\{ - \frac{g(t)}{2} - h(t) \}$$ (3.12)

<u>Proof</u>: We proceed to find the auto-correlation function of the waveform,

b(u), by using the characteristic function approach of Appendix A.

When b(u) = exp { jz(u)}, with <u>z</u> Gaussian,

$$R_b(t) \triangleq E\{ b(u+t) b^*(u) \}$$

$$R_b(t) = F_{\underline{z}}(+1, -1)$$ (3.13)

where $F_{\underline{z}}$ is the characteristic function of <u>z</u>, i.e. for a Gaussian random

vector, <u>z</u>, with covariance matrix, $M_{\underline{z}}$

$$F_{\underline{z}}(\underline{p}) = \exp\{ - \frac{1}{2} \underline{p}^T M_{\underline{z}} \underline{p} \}$$

If The covariance matrix elements are m_{ij}, then, for $\underline{p} = [1, \quad -1]^T$

$$R_b(t) = \exp \{ - \frac{1}{2} [m_{11} - 2m_{12} + m_{22}] \}$$

When

$$M_{\underline{z}} = \begin{bmatrix} a(u) + g(t) & a(u) - h(t) \\ a(u) - h(t) & a(u) \end{bmatrix}$$ (3.14)

we have finally

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$$R_b(t) = \exp\left\{-\frac{g(t)}{2} - h(t)\right\} \qquad (3.15)$$

Note that although z(u) is non-stationary, the auto-correlation function of the oscillator rf signal, b(u), depends only on t for all u.

<div align="right">Q.E.D.</div>

Next, we investigate the ergodic properties of our phase models.

3.2. Ergodicity:

It is well known [3.1], [3.2] that if a process is ergodic, then time averages are equal to the corresponding ensemble averages. The properties of the process can therefore be determined from one sample function which is useful from the practical standpoint.

In this section we establish a number of theorems on ergodicity for three statistical parameters of the rf oscillator signal; the mean, the auto-correlation function, and the power spectral density. We consider all the three phase random processes; the white phase, random walk phase, and the random walk frequency.

3.2.1. Ergodicity in the Mean

A test for ergodicity in the mean of a random process is stated as follows [3.2], pp 328

Theorem:

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} b(u)\, du = E(b(u)) = \eta \qquad (3.16)$$

iff

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{2T} \left(1 - \frac{|t|}{2T} \right) [\, R_b(t) - \eta^2 \,]\, dt = 0 \qquad (3.17)$$

where $R_b(t) = E[b(u+t)\, b^*(u)]$ is the auto-correlation function of $b(u)$.

The mean, η, and the auto-correlation function, $R_b(t)$, of the phase random processes have been determined in Appendix D. These are required for the investigation of ergodicity in the mean. First we consider the white (or "near white") phase process.

Theorem 3:

The rf signal process due to "near white" phase instability is ergodic in the mean if the random phase at any time, u, approaches complete decorrelation after some time shift, t, i.e. for

$$0 < T_o / 2 < |t| < \infty$$

the auto-correlation function satisfies

$$R_y(t) \triangleq E(y(u+t)\, y^*(u)) = 0 \qquad (3.18)$$

Proof: Using appendix A and appendix B it can be shown from the characteristic function that for the white phase process, $y(u)$, with

$|t| > T_o / 2$, the auto-correlation of the rf signal, $b(u)$, is

$$E(\exp\{ j[y(u+t)-y(u)] \}) = F_{\underline{y}}(1, -1) = \exp\{ -\frac{1}{2}(m_{11} + m_{22}) + m_{12} \}$$

$$= R_b(t) = \exp(-o^2) \qquad\qquad t > \frac{T_o}{2} \qquad (3.19a)$$

Also from the characteristic function approach it is easy to show that

$$F_{\underline{y}}(1) = \exp\{ -\frac{1}{2} m_{11} \}$$

$$\eta_b = E(\exp(jy(u))) = \exp(-\sigma_o^2 \frac{c_o}{2}) \qquad\qquad (3.19b)$$

Substitution of equation (3.19) into (3.17) gives

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(-\sigma_o^2) - \exp(-\sigma_o^2 c_o)] \, dt = 0 \quad : \quad c_o = 1$$

$$(3.20)$$

$$= \exp(-\sigma_o^2) - \exp(-\sigma_o^2 c_o) \neq 0 \quad : \quad c_o \neq 1$$

i.e., the rf signal process is ergodic in the mean if for some $t > T_o/2$, $c_o - 1 = 0$. The covariance matrix for this condition is

$$E(\underline{y}\underline{y}^T) = \sigma_o^2 \begin{bmatrix} c_o & c_o - 1 \\ c_o - 1 & c_o \end{bmatrix} = \sigma_o^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad : \quad T_o/2 < |t| < \infty \quad (3.21)$$

Since the off diagonal terms denote correlation with respect to the time

shift, t, then $c_o = 1$ implies that the phase process, $y(u_o)$, has become decorrelated at time $u+t$. Hence,

$$c_o = 1 \implies R_y(t) = 0 \quad : \quad |t| > T_o/2. \quad \text{Q.E.D.}$$

If a correlated component of the phase at time, u, does exist, it must be known or deterministic in order for ergodicity of the mean to apply.

Theorem 4:

The rf signal process, b(u), due to the *random walk phase* instability *process* is *ergodic in the mean* if the reference phase at time, u, is uniformly distributed phase over 2π; c.f., Theorem 1, Lemma 1

Proof: It has been shown in the appendices that for the random walk phase process, x(u),

$$R_b(t) = \exp(-\sigma_1^2 \frac{t}{2}) \qquad t > 0 \qquad (3.22a)$$

Using the characteristic function in conjunction with the arbitrary phase, we can evaluate

$$\eta_b = \exp(-\sigma_1^2 c_1/2) \qquad (3.22b)$$

Substituting equation (3.22) into equation (3.17) gives

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{t}{2T}\right)\left[\exp\left(-\sigma_1^2 \frac{t}{2}\right) - \exp(-\sigma_1^2 c_1)\right] dt = -\exp(-\sigma_1^2 c_1)$$

$$\neq 0 \qquad : \quad c_1 < \infty$$

$$= 0 \qquad : \quad c_1 = \infty \qquad\qquad (3.23)$$

This completes the proof for the random walk phase.   Q.E.D.

Theorem 5:

The rf signal, $b(u)$, due to the random walk frequency process, $\dot{v}(u)$, is ergodic in the mean if the reference phase at time, u, is uniformly distributed over 2π, cf. lemma 1.

Proof: For the "Random Walk" Frequency, $\dot{v}(u)$, it can be shown from the appendices that for the process, $\dot{v}(u)$,

$$R_b(t) = \exp\left(-\sigma_2^2 [3c_2 + t] t^2 /6\right) \qquad\qquad (3.24a)$$

Using the characteristic function in conjunction with the arbitrary phase variance we find the mean of the rf signal

$$\hbar_b = \exp(-\sigma_2^2 c_3 /2) \qquad\qquad (3.24b)$$

Substituting equation (3.24) into equation (3.17) gives

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(-\sigma_2^2 [3c_2 + t]t^2/6) - \exp(-\sigma_2^2 c_3^2/2)] \, dt$$

$$= -\exp(-\sigma_2^2 c_3^2/2) \neq 0 \qquad : \quad c_3 < \infty$$

$$= 0 \qquad : \quad c_3 = \infty \qquad (3.25)$$

$$\text{Q.E.D.}$$

This completes the proof for the random walk frequency process, $v(u)$. We now turn our attention to ergodicity of the auto-correlation function.

3.2.2. Ergodicity in Auto-correlation Function

The three phase instability models, white phase, random walk phase, and random walk frequency will now be considered for the ergodicity in auto-correlation function of the oscillator rf signal.

A test for ergodicity in the auto-correlation function of a random process is stated as follows [3.2] (pp 330).

Theorem: For a given s

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} b(u+s)b^*(u) \, du = E\{b(u+s)b^*(u)\} = R_b(s) \qquad (3.26)$$

iff

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[R_{\mathcal{B}}(t) - R_b^2(s)] \, dt = 0 \qquad (3.27)$$

where $R_b(s) = E[b(u+s)b^*(u)]$ is the auto-correlation function and

$R_{\phi\phi}(t) = E[\, b(u+s+t)b^*(u+t)b^*(u+s)b(u)\,]$ is the fourth order moment of

the rf signal, b(u). We will proceed to establish theorems relating to

the ergodicity of the auto-correlation function of the process

$$b(u) = \exp(jz(u)).$$

To accomplish this we need to determine the second and the fourth order

moments appearing in equation (3.27). We will use the procedure and

results given in appendix D to determine these moments.

Theorem 6:

If the phase random process of an rf signal, b(u), is a <u>white noise</u>

<u>phase process</u>, y(u), then <u>b(u) is ergodic in its auto-correlation</u>

<u>function</u>.

Proof: With appropriate substitutions of $R_{\phi\phi}(t)$ and $R_b(s)$ from

Appendix D, for the white phase process, equation (3.27) is evaluated

under three regions for the shift variables, t and s i.e.

     1) $0 < t < s + T_o / 2$: all s

and   2) $0 < s + T_o / 2 < t$: $s < T_o / 2$

     3) $0 < s + T_o / 2 < t$: $s > T_o / 2$

The integral, equation (3.27), for the first region is finite due to the

fact that the area under the white (or "near white") impulse type

function is finite for a finite interval, T_o .

For the second region, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(- \sigma_o^2 \frac{4}{T} s) - \exp(- \sigma_o^2 \frac{4}{T} s)] \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[0] \, dt \doteq 0 \qquad (3.28a)$$

$$: \; s < T_o/2 < t-s$$

and for the third region, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(- 2\sigma_o^2) - \exp(- 2\sigma_o^2)] \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[0] \, dt \doteq 0 \qquad (3.28b)$$

$$: \; T_o/2 < s < t-T_o/2$$

We see that the limit of equation (3.27) approaches zero for the three conditions above. Hence, equation (3.27) is satisfied for all s.

<div align="right">Q.E.D.</div>

We conclude that the auto-correlation function of the rf signal, b(u), is ergodic when the phase instability is a white (or near white) random process, y(u).

Theorem 7:

If the phase random process is a random walk pr..ess, x(u), then the rf signal, b(u), is ergodic in auto-correlation function.

Proof: The appropriate second and fourth order moments for the random walk phase are taken from appendix D and are substituted into equation (3.27).

It can be argued when t is such that $0 < t < s$, the integral of equation (3.27) is finite and so equation (3.27) approaches zero. For all other values of t such that $0 < s < t$, the integral is identically zero. That is

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(-\sigma_1^2 s) - \exp(-\sigma_1^2 s)] \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[0] \, dt = 0 \qquad (3.29)$$

Therefore, equation (3.27) is satisfied. $Q.E.D.$

We conclude that the the rf signal, b(u), is ergodic in auto-correlation function of the when the phase random process is a random walk phase process, x(u).

Theorem 8:

If the phase random process is a _random_ _walk_ _frequency_ _process,_ v(u), then the rf signal, _b(u),_ _is_ _not_ _ergodic_ _in_ _auto-correlation_ _function._

Proof: For random walk (Brownian Motion) frequency instabil.y, equation (3.27) uses the fourth and second order moments obtained in equation (D.7). The result is

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[\exp(- \sigma_2^2 \frac{s}{6}[3t-s]) - \exp(- \sigma_2^2 \frac{s}{3}[3c_2+s])] \, dt$$

The first term in the brackets has a negative exponent and will bring the limit to zero. The second term is a constant which leads to

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} (1 - \frac{t}{2T})[-\exp(- \sigma_2^2 \frac{s}{3}[3c_2+s])] \, dt$$

$$= -\exp(- \sigma_2^2 \frac{s}{3}[3c_2+s]) \neq 0 \qquad\qquad (3.30)$$

We conclude that the rf signal, $b(u)$, is not ergodic in auto-correlation function of the when the phase random process is a random walk frequency process, $v(u)$.   Q.E.D.

Observe the rf signal, $b(u)$, is not ergodic in auto-correlation function when the phase random process consists only of a long term drift offset frequency. This is obvious due to the presence of c_2 and σ_2 in equation (3.30).

For certain processes, note that ergodicity may exist in the auto-correlation function and yet may not exist in the mean. This is easily seen through the fact that the mean phase, though random and non-ergodic across the ensemble, is subtracted out for any one sample function in the process of finding the auto-correlation function. This is done through the functional operation

$$R_b(t) = E(b(u+t)b^*(u)) = E(\exp[jz(u+t)-jz(u)])$$

eliminating any effects which the mean phase variable has on the rf auto-correlation function.

This effect is brought out in the next theorem on ergodicity of the power spectral density. In this case the specular component and spread component are not necessarily mutually ergodic.

3.2.3. Ergodicity in Power Spectral Density

Many references [3.2], [3.7] provide conditions for determining the power spectrum from the average random power,i.e. the time average. When these conditions are met, then the limiting value of the time average is equal to the power spectral density, i.e.

$$S(w) = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{T} x(u) \exp(-jwu) \, du \right|^2$$

and the power spectral density is ergodic.

Ergodicity of the power spectral density,

$$S(w) \triangleq \int_{-\infty}^{\infty} R(t) \exp(-jwt) \, dt$$

requires first, that the expected value of the "average random power",

$$S_T(w) \triangleq \frac{1}{2T} \left| \int_{-T}^{T} b(u) \exp(-jwu) \, du \right|^2$$

should tend to S(w) in the limit; and second, that its variance should

- 66 -

tend toward zero in the limit, $T \longrightarrow \infty$. We will only address the first of these requirements which will provide us with a necessary condition for testing ergodicity in power spectral density. We proceed, using the theorem provided on page 343 of [3.1] which follows

Theorem: If

$$\int_{-\infty}^{\infty} |tR_b(t)| \, dt < \infty \qquad (3.31)$$

then

$$\lim_{T \to \infty} E\{S_T(w)\} = S(w) = \int_{-\infty}^{\infty} R_b(t) \exp(-jwt) \, dt \qquad (3.32)$$

where

$$S_T(w) = \frac{1}{2T} \left| \int_{-T}^{T} x(u) \exp(-jwu) \, du \right|^2$$

and

$$R(t) = E\{x(u+t)x^*(u)\}.$$

That is, if equation (3.31) is satisfied, then a necessary condition has been met for the power spectral density of the process, $b(u)$, to be equal to the time average of the power spectrum obtained from absolute value squared of the Fourier transform of the sample function. We now state our theorem for ergodicity of Power Spectral Density.

Theorem 9:

If a random phase process, z(u), consists of the sum of a white Gaussian, finite variance, phase process and higher order independent increment processes, then the corresponding power spectral density is not ergodic. The random correlated component exists over the interval, $-\infty \leq u \leq \infty$, and has a finite Gaussian variance, σ^2(white), i.e. not uniformly distributed over modulo 2π, c.f. Lemma 1.

Proof: It has been established, in our work thus far and elseware [3.3], [3.8], [3.9] that the auto-correlation function of an oscillator rf signal containing white, random walk phase, and random walk frequency components is of the form

$$R_b(t) = \exp(-\sigma_o^2 \frac{2}{T_o}|t| - b|t| - c|t|^2 - d|t|^3) \qquad : |t| \leq T_o/2$$

(3.33)

$$R_b(t) = \exp(-\sigma_o^2 - b|t| - c|t|^2 - d|t|^3) \qquad : |t| > T_o/2$$

Clearly, the exponential terms containing $|t|$ and powers there-of force the integral in equation (3.31) to converge to some finite value so that equation (3.32) is satisfied. The only condition we will investigate is when $b = c = d = 0$, i.e. the random walk phase, the offset frequency, and the random walk frequency component are absent (or of zero level). Then only the white phase component is present. We have for this case

$$R_b(t) = \exp(-\sigma_o^2 \frac{2}{T_o} |t|) \quad : \quad |t| \leq T_o/2$$

$$R_b(t) = \exp(-\sigma_o^2) \quad : \quad |t| > T_o/2$$

Defining

$$f(t) \triangleq \text{rect}\{t/T_o\} \, [\, \exp(-\sigma_o^2 \frac{2}{T_o} |t|) - \exp(-\sigma_o^2) \,]$$

where

$$\text{rect}\{t\} \triangleq 1 \quad : \quad |t| \leq 1/2$$

$$\triangleq 0 \quad : \quad \text{otherwise}$$

we can write the above relationships as follows:

$$R_b(t) = f(t) + \exp(-\sigma_o^2) \qquad (3.34)$$

Substituting equation (3.34) into equation (3.31), we have

$$\int_{-\infty}^{\infty} \left| t[f(t) + \exp(-\sigma_o^2)] \right| \, dt = \int_{-\infty}^{\infty} \left| t \, f(t) \right| \, dt + \int_{-\infty}^{\infty} \left| t \, \exp(-\sigma_o^2) \right| \, dt$$

$$(3.35)$$

Since $f(t)$ is bounded and is non-zero for finite duration only, the first integral on the right of equation (3.35) is finite and therefore satisfies the necessary condition stated in equation (3.31). It remains to determine the effects which the second integral on the right of equation (3.35) have on the ergodicity of the power spectral density.

This integral arises from that term of the "near white" phase

component, equation (3.34), which contributes the constant, $\exp\{-\sigma_o^2\}$, to the auto-correlation function for all $T_o/2 \leq |t| \leq \infty$. Following through with the evaluation of the second integral, in equation (3.35) (equation $(10-27)$ of $[3.2]$) for the pedestal, $\exp(-\sigma_o^2)$, we obtain

$$\lim_{T \to \infty} E[S_T(w)] = \lim_{T \to \infty} \exp(-\sigma_o^2) \frac{1 - \cos(2Tw)}{T^2 w}$$

As $T \to \infty$, this becomes an impulse function in w, ($[3.9]$, pp 378, equation 633). We can write

$$\lim_{T \to \infty} [S_T(w)] = 2\pi \exp(-\sigma_o^2) d(w) \tag{3.36}$$

where $d(w)$ is an impulse function.

We see how the pedestal component, i.e. the constant auto-correlation function for all t, of the "near white" phase process gives rise to the specular component, or the frequency component representing the correlated random part of the oscillator. The pedestal component is not ergodic in power spectral density. The spectral spreading due to the time decorrelated component, the first term in equation (3.35), does however satisfy the condition for ergodicity given in the above theorem. Hence, theorem 9 is proved. These properties are shown in figure 3.2.

Theorems have been provided for stationarity and ergodicity of the rf signal, $b(u)$, when the phase random process is white, random walk, and/or when the frequency random process is random walk. Properties covered were the mean, auto-correlation function, and power spectral density. Table 3.1 summarizes the results.

- 70 -
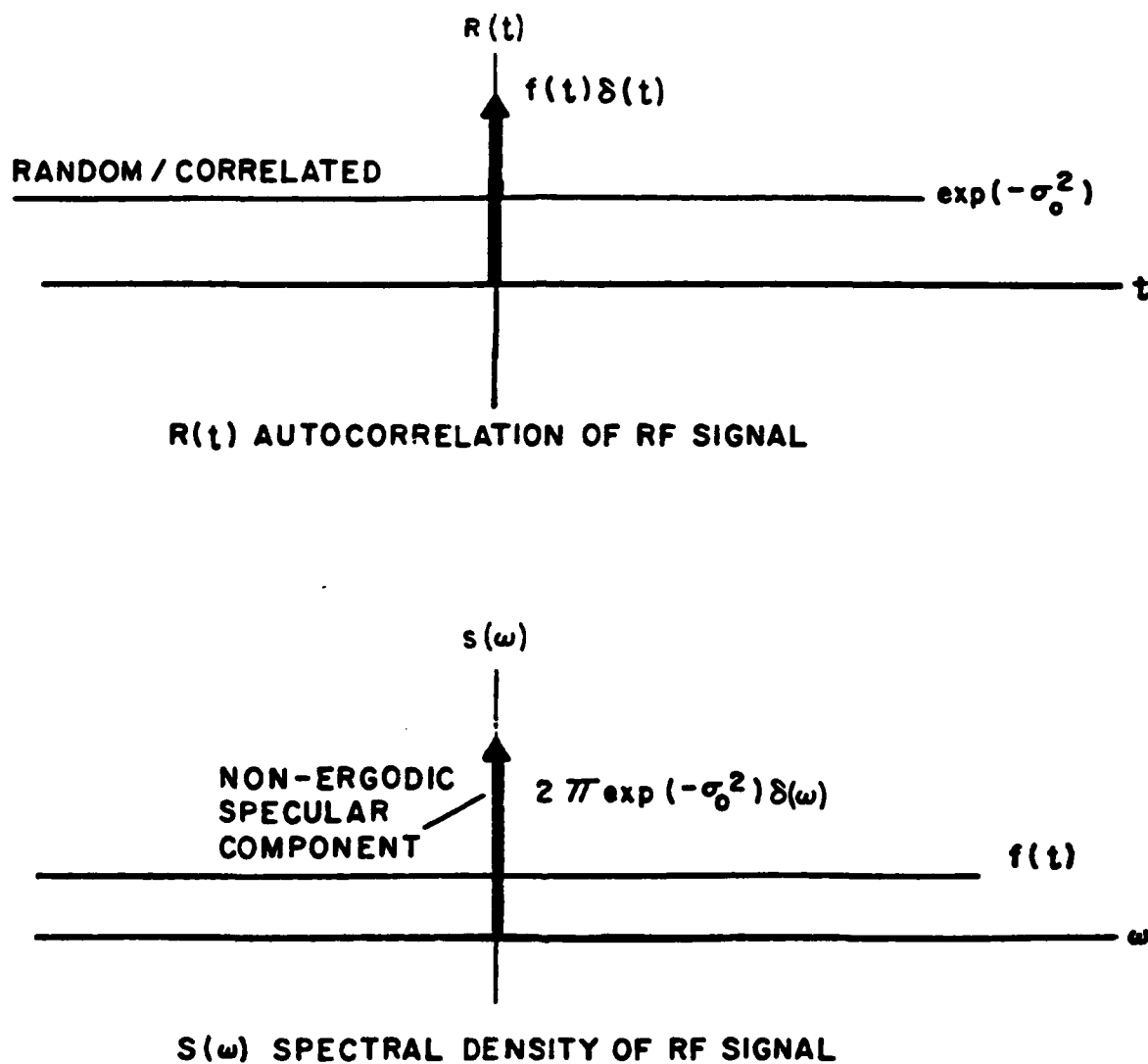
FIGURE 3.2   WHITE PHASE INSTABILITY PROPERTIES

TABLE 3.1   CONDITIONS FOR ERGODICITY

PROCESS	MEAN	AUTO-CORRELATION	SPECTRUM
y(u) WHITE	ERGODIC: RANDOM COMPONENT OF STARTING PHASE TENDS TOWARD COMP-LETE DECORRELATION AT u < ∞	ERGODIC: FOR STARTING PHASE HAVING ANY pdf	POSSIBLY ERGODIC: FOR SPECTRAL SPREAD COMPONENT ONLY. NOT ERGODIC: FOR SPECTRAL SPECULAR DUE TO RANDOMNESS
x(u) RANDOM WALK PHASE	ERGODIC: STARTING PHASE HAS UNIFORM pdf	ERGODIC: FOR STARTING PHASE HAVING ANY pdf	POSSIBLY ERGODIC: ALL CONDITIONS
v(u) RANDOM WALK FREQUENCY	ERGODIC: STARTING PHASE HAS UNIFORM pdf	NOT ERGODIC: BY VIRTUE OF STARTING & RW FREQUENCY COMPONENT	POSSIBLY ERGODIC: ALL CONDITIONS

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IV   SPECTRAL SPREADING DUE TO PHASE INSTABILITY

4.0.   Introduction

It is well known that a monochromatic oscillator signal provides a specular (impulse) mapping into the spectrum domain.   Such a signal, when modulated by an envelope waveform, fo(u), provides a frequency domain convolution of the specular spectrum with the envelope sprectrum to result in the modulated system spectrum.   Normally, the ideal envelope spectrum is characterized by true nulls, of infinite depth, in the frequency domain.   When convolved with a frequency specular the spectrum of the modulated system waveform reproduces these infinitely deep nulls.   With such an ideal spectrum the potential exists, theoretically, to process a second signal located at one of these nulls to allow almost perfect discrimination between the two signals when only low level thermal noise is present.   A simple example of this situation is found in radar systems.   When the transmitted signal of a ground based radar is such a modulated waveform the return from a moving target is doppler shifted relative to the return from terrain clutter.   By observing the return within the nulls of the unshifted spectrum one can detect the moving target against clutter with a level of performance limited only by the thermal noise of the system.

With the introduction of phase instability the spectrum of the oscillator is no longer a specular impulse.   The specular broadening effect due to the oscillator instability causes a spectral spreading of the system modulated waveform with an associated deterioration of the null depths.   As the potential discrimination in the null region degrades due to phase instability which is in addition to thermal noise, the system performance suffers further.   As mentioned earlier, we only

consider the effect of phase instability in this dissertation. The degradation is interrelated with a variety of system parameters such as the observed spectral region; the level and type of instability, i.e. white, Wiener, etc.; and the type of envelope waveform used to modulate the oscillator. We will discuss the interrelation between these parameters and their spectral spreading effects which deteriorate the observable nulls.

Next, we will describe the modulation system and the operations involved. The modulated waveform, denoted by fk(u) in equation (1.2), (henceforth denoted by f(u)) with average auto-correlation function Rf(t) in equation (1.4), (henceforth denoted by R(t)) is often perceived s a train of pulses as shown in figure 4.1. The waveform, f(u), is the output of the modulator shown in figure 1.1a and its average auto-correlation function, R(t) is produced by the equivalent diagram of figure 1.1b. The waveform, f(u), consists of a stationary random process, b(u), driven by the non-stationary random phase process, z(u), shown in figure 4.2a.
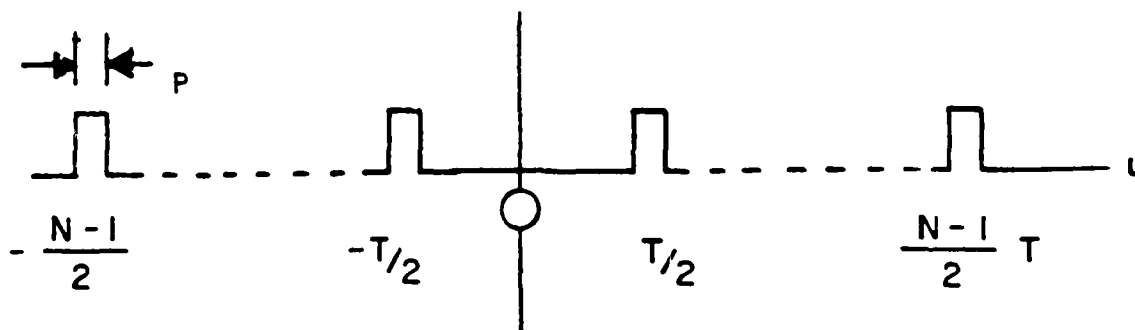
FIGURE 4.1   ASSUMED PULSE TRAIN

The process is multiplied by some ideal deterministic waveform, for example, a finite train of N narrow pulses, fo(u), shown in figure 4.2b. This is generally a non-stationary random process which produces the pulse train, f(u), whose amplitude and random phase is shown in figure 4.2c. Finally this pulse train is convolved with the pulse shape, so(u), shown in figure 4.2d to produce the desired train, s(u), shown in figure 4.2e. If the pulse shape were rectangular it would produce the pulse train shown in figure 4.1. Once again in this chapter, we shall treat white phase and random walk phase models seperately from the random walk frequency model. Only analytical results are obtained. Numerical results will be treated in the next chapter.

4.1. White Phase and Random Walk Phase

As indicated earlier, for this model the oscillator signal, b(u), includes a non-stationary phase random process, z(u), which is the sum of a "near white" process, y(u), and an independent Wiener (BM or random walk) process, x(u).

In this section, we shall treat the following three modulating envelope waveforms:

1) A cw waveform as given in equation (2.5a);

2) An infinite train of narrow pulses, d(u-iT);

3) A finite train of N short pulses as given in equation (2.5b).

These waveforms are shown in figures 4.3a, 4.3b, and 4.3c respectively.

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Z(u)

PHASE INSTABILITY RANDOM PROCESS
FIGURE (a)

$f_o(u)$

$\frac{N-1}{2} T$ $-T/2$ $T/2$ $\frac{N-1}{2} T$

2N MODULATING PULSE TRAIN $f_o(u)$
FIGURE (b)

f(u)   AMPLITUDE

u

PHASE

u

PULSE TRAIN AMPLITUDE AND PHASE f(u)
FIGURE (c)

P

SINGLE PULSE SHAPE $S_o(t)$

FIGURE (d)

S(u)   AMPLITUDE

$-\frac{N-1}{2} T$ $-\frac{1}{2} T$ $\frac{1}{2} T$ $\frac{N-1}{2} T$

PHASE

u

ACTUAL PULSE TRAIN
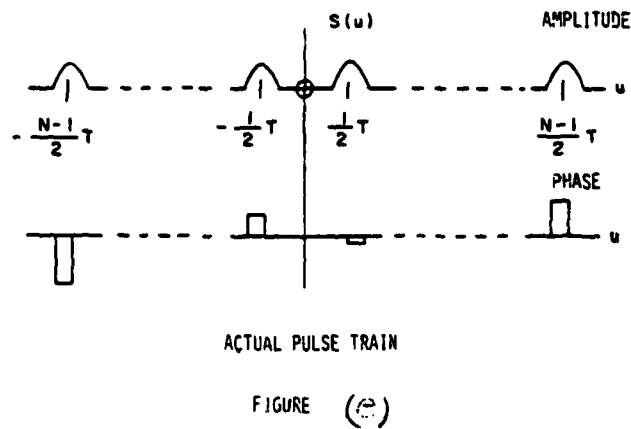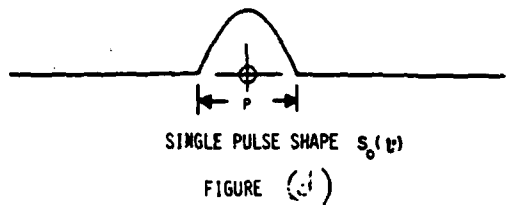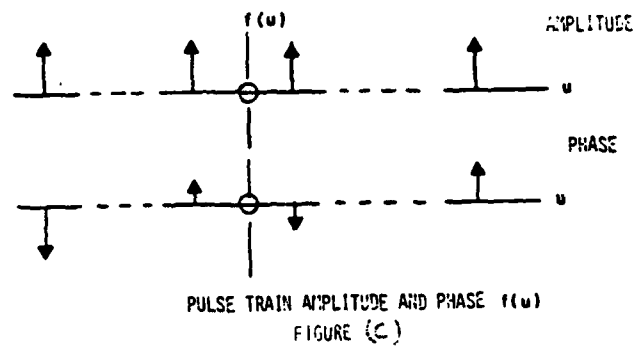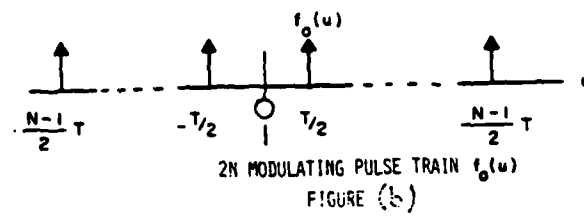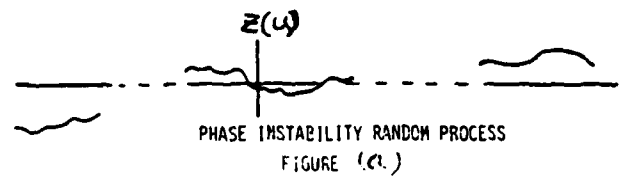
FIGURE (e)

FIGURE 4.2.   MODULATION OF AN OSCILLATOR RF SIGNAL

- 77 -
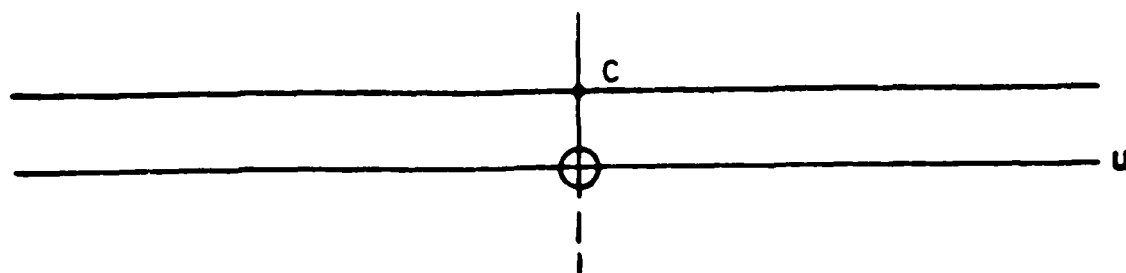
CW WAVEFORM COMPLEX EQUIVALENT

a
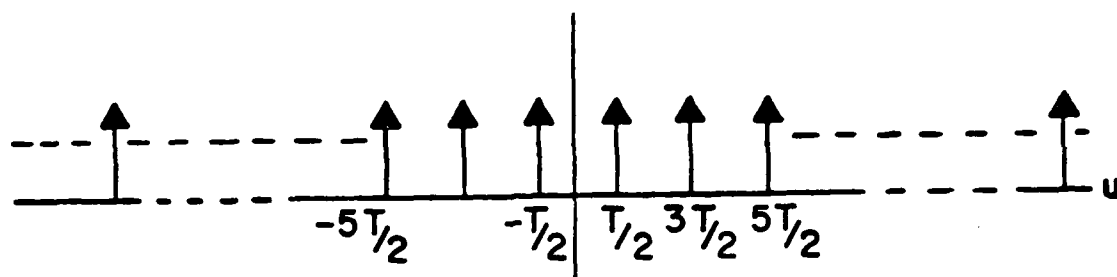
INFINITE TRAIN SHORT PULSES

b
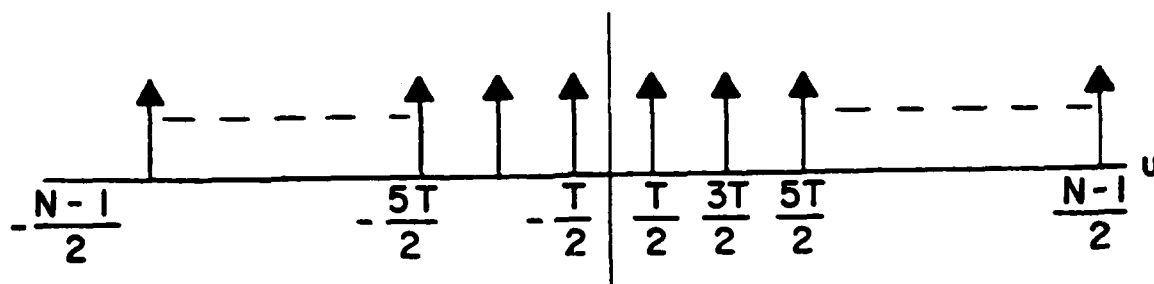
FINITE TRAIN SHORT PULSES

c

FIGURE 4.3   ENVELOPE WAVEFORMS

For the cw waveform and the infinite train of short pulses, we   use   the
time average auto-correlation function defined as:

$$R_o(t) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} f_o(u) f_o^*(u-t) \, du \qquad (4.1a)$$

For the finite pulse train of short pulses, we use the time average auto-correlation function defined as:

$$R_o(t) = \int_{-\infty}^{\infty} f_o(u) f_o^*(u-t) \, du \qquad (4.1b)$$

Multiplying equation (4.1a) and equation (4.1b) by the auto-correlation function of the oscillator rf signal, $R_b(t)$, and taking the Fourier transform of the product results in the average power spectrum and average energy spectrum respectively [4.1]. The average power spectrum becomes power spectral density only when the modulating envelope waveform, $f_o(u)$, is stationary, e.g. a cw.

From equation (1.4), we can obtain the average auto-correlation function of our system output wav forn,

$$R(t) = R_b(t) R_o(t) \qquad (4.2).$$

Recall that the system output waveform is generated by taking the following product, equation (1.2)

$$f(u) = f_o(u) b(u) \qquad (4.3).$$

The general procedure that we shall follow for each of the three waveforms is:

    1) Evaluate, the oscillator auto-correlation function, $Rb(t)$,

using the procedures developed in chapter two, equations (2.10) and (2.15).

ii) Evaluate the time average auto-correlation function, $R_o(t)$, of the modulating envelope waveform, $f_o(u)$, using equation (4.1).

iii) Determine the average auto-correlation function, $R(t)$ of the system output using equation (4.2) by substituting $R_b(t)$ and $R_o(t)$ found in (i) and (ii) above.

iv) Find the average power/energy spectrum through the Fourier transform [4.1], [4.2].

$$S(w) = \int_{-\infty}^{\infty} R(t) e^{-jwt} dt$$

4.1.1. The cw envelope waveform:

This waveform, $f_o(u) = c$, is a stationary random process. The av rage auto-correlation function of the output waveform, $f(u)$, becomes the auto-correlation function and the average power spectrum becomes the power spectral density [4.1]. For the cw envelope waveform, the time average auto-correlation function is the auto-correlation function (ensemble av rage) and is given by:

$$R_o(t) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} c \, c^* \, dt = |c|^2 \qquad (4.4).$$

Using equation (4.2) we determine the auto-correlation function of the modulated output waveform as follows:

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$$R(t) = R_o(t)\, R_b(t)$$

$$R(t) = |c|^2 \exp\left\{-\frac{1}{2}\left(\sigma_1^2 + 2\,\sigma_o^2\,\frac{2}{T_o}\right)|t|\right\}$$

$$= |c|^2 \exp\left\{-k_o\,\frac{2}{T_o}\,|t|\right\} \qquad : \ |t| < \frac{T_o}{2}$$

$$R(t) = |c|^2 \exp\left\{-\frac{1}{2}\left(\sigma_1^2\,|t| + 2\,\sigma_o^2\right)\right\}$$

$$= |c|^2 \exp\left\{-\left(k_1\,\frac{2}{T_o}\,|t| + \sigma_o^2\right)\right\} \qquad : \ |t| > \frac{T_o}{2} \qquad (4.5a)$$

where $\quad k_o = \dfrac{\sigma_1^2\,T_o}{4} + \sigma_o^2 \quad$ and $\quad k_1 = \dfrac{\sigma_1^2\,T_o}{4}$.

The shape of this function is shown in figure 2.2.

A generalization of the above equation can be made to include other shapes for the impulse type auto-correlation function. By simply introducing a superposition of any number, say (L-1), of "near white" phase processes, each having different triangular slopes and decorrelation times, see figure 4.4, we can extend the stationary "near white" phase model, $y(u)$, to include a variety of auto-correlation functions consisting of concave curvatures. It is not difficult to show that the expression for these extended cases can be written as

$$R(t) = |c|^2 \exp\left\{-\left(k' + k_1\,|t|\right)\right\} \qquad (4.5b)$$

where $i = 0, 1, 2, \ldots (L-1), L$; $T_{-1} = 0 < T_i < T_L = \infty$, and for

$$\frac{T_{i-1}}{2} < |t| \le \frac{T_i}{2} \quad , \quad k'_{i-1} = \sum_{j=0}^{i-1} \sigma_j^2 \quad , \quad k_i = \frac{\sigma_L^2}{2} + \sum_{j=1}^{L-1} \sigma_j^2 \frac{2}{T_j}$$

and the T'_is are the decorrelation times of the additive "near white" Gaussian phase random processes. The logarithm of equation (4.5b) is plotted in figure 4.4.

$$\ln R_f(t) = -(K'_{i-1} + K_i |t|)$$
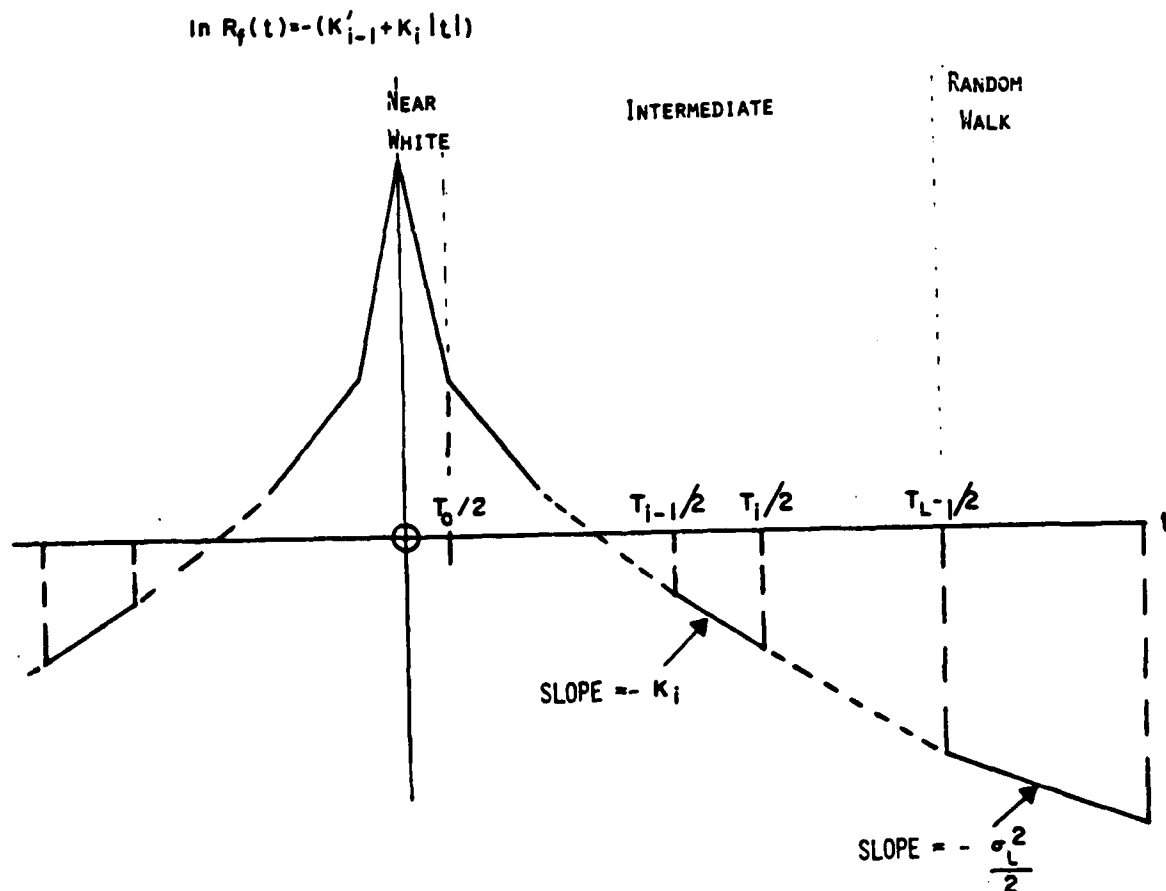
FIGURE 4.4. LOGARITHM – OSCILLATOR PHASE AUTO–CORRELATION FUNCTION FOR NEAR WHITE AND RANDOM WALK GENERALIZED PHASE MODEL

Taking the Fourier transform of equation (4.5a), we have the power
spectral density of the cw waveform.

$$S(w) =$$

$$|c|^2 T_o \left[\frac{k_o}{k_o^2 + b^2} - \exp\{-k_o\} \left[\frac{k_o \cos b - b \sin b}{k_o^2 + b^2} - \frac{k_1 \cos b - b \sin b}{k_1^2 + b^2} \right] \right]$$

$$(4.6a)$$

where $k_o = \dfrac{\sigma_1^2 T_o}{4} + \sigma^2$, $k_1 = \dfrac{\sigma_1^2 T_o}{4}$, and $b = w \dfrac{T_o}{2}$.

The power spectral density of the generalized model, equation
(4.5b), which contains the superposition of the "near white" stationary
phase random processes and the random walk phase random process is
found by taking its transform.   The result is

$$S(w) = |c|^2 \sum_{i=0}^{L} \left\{ \frac{\exp(-k'_{i-1})}{k_1^2 + w^2} \left[\exp\{-k_i \frac{T_{i-1}}{2}\} [k_1 \cos(w \frac{T_{i-1}}{2}) - w \sin(w \frac{T_{i-1}}{2})] \right. \right.$$

$$\left. \left. -\exp\{-k_i \frac{T_i}{2}\} [k_1 \cos(w \frac{T_i}{2}) - w \sin(w \frac{T_i}{2})] \right] \right\}$$ $$(4.6b)$$

where k'_{i-1} , k_1 , and T_i are defined after equation (4.5b).

4.1.2. The infinite pulse train:

The time average auto-correlation function of this envelope waveform
is given by:

$$R_o(t) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} c \sqrt{\frac{T}{\Delta t}} \sum_{-\infty}^{\infty} d(u+t-\ell T) \; c \sqrt{\frac{T}{\Delta t}} \sum_{-\infty}^{\infty} d(u-jT) \, du$$

where T is the interpulse period and $\Delta t \ll 1$ is the pulsewidth. This waveform has been normalized so that the average energy is $|c|^2$.

Setting $i = \ell - j$ and carrying out the integration, we obtain

$$R_o(t) = |c|^2 \frac{T}{\Delta t} \sum_{i=-\infty}^{i=\infty} d(t - iT) \qquad (4.7)$$

For the modulated output waveform, we let m/2 : m even; denote the number of interpulse intervals, T, over which the white stationary term is still partially correlated and we allow decorrelation to occur within the next interpulse period. Then the stationary "near white" decorrelation interval

$$-\frac{T_o}{2} \le t \le \frac{T_o}{2}$$

will be related to the interpulse interval, T, by

$$mT < T_o < (m + 2) T$$

This relationship is illustrated in figure 4.5a. The corresponding average auto-correlation function for the modulated waveform is obtained by multiplying $R_o(t)$ of equation (4.7) with $R_b(t)$ of equation (2.10) and is given by:

$$R(t) = |c|^2 \frac{T}{\Delta t} \sum_{i=-m/2}^{i=m/2} \exp\{-\frac{1}{2}(\sigma_1^2 + 2\sigma_o^2 \frac{2}{T})|t|\} \, d(t-iT)$$

$$: \quad |t| < To/2$$

$$R(t) = |c|^2 \frac{T}{\Delta t} \sum_{\substack{i=-\infty \\ i \notin [-m/2, m/2]}}^{i=\infty} \exp\{-\frac{1}{2}(\sigma_1^2 |t| + 2\sigma_o^2) \, d(t-iT) \qquad (4.8)$$

$$: \quad |t| > To/2$$

Next we determine the average power spectrum of the output system waveform due to the infinite pulse train. Here we do not consider the generalized phase model containing the superposition of "near white" phase random processes introduced in equation (4.5b), figure 4.4. The average power spectrum for such a generalized process can, however be found in the same manner. Figure 4.5a shows the average auto-correlation function of the product of the infinite pulse train and the oscillator wav form. The average power spectrum is given by:

$$S(w) = |c|^2 T \left[1 + \sum_{i=1}^{m/2} \exp\{-ik_o T\}[\exp\{jiwT\} + \exp\{-jiwT\}] \right.$$

$$\left. + \exp\{-(k_o - k_1)\frac{To}{2}\} \sum_{i=(m/2+1)}^{\infty} \exp\{-ik_1 T\}[\exp\{jiwT\} + \exp\{-jiwT\}] \right]$$

FIGURE 4.5   AVERAGE AUTO-CORRELATION FUNCTION OF PRODUCT:
FINITE PULSE TRAIN AND OSCILLATOR RF SIGNAL

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$$S(w) = |c|^2\, T \left\{ \frac{\sinh a_o - \exp\{-\tfrac{m}{2} a_o\}\left[\cos(\tfrac{m}{2}+1)b - \exp\{-a_o\}\cos(\tfrac{m}{2} b)\right]}{\cosh a_o - \cos b} \right.$$

$$\left. + \frac{\exp\{-(\sigma_o^2 + \tfrac{m}{2} a_1)\}\left[\cos(\tfrac{m}{2}+1)b - \exp\{-a_1\}\cos(\tfrac{m}{2} b)\right]}{\cosh a_1 - \cos b} \right\} \qquad (4.9a)$$

where $a_o = k_o T$, $b = wT$, $k_o = \dfrac{1}{2} + \sigma_o^2\, \dfrac{2}{T_o}$, $k_1 = \dfrac{\sigma_1^2}{2}$, and $a_1 = k_1 T$

For the special case, when the near white stationary phase term of figure 2.1a becomes decorrelated immediately after the first pulse, i.e. when

$$T_o \ll T \quad \text{or} \quad m = 0^+$$

as shown in figure 4.5b, the expression (given in equation (4.9a)) for average power spectrum of the infinite pulse train reduces to

$$S(w) = |c|^2\, \frac{T}{\Delta t}\left[1 + \exp\{-\sigma_o^2\}\, \frac{2\cos b - \exp\{-a\}}{\cosh a - \cos b}\right] \qquad (4.9b)$$

where $a = k_1 T$

4.1.3. The finite pulse train of N pulses

The time average auto-correlation function of the envelope waveform is:

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$$R_o(t) = \left[\int_{-([N-1]T/2 + t)}^{[N-1]T/2} + \int_{-[N-1]T/2}^{[N-1]T/2 - t} \right] \times$$
$$\qquad\qquad\qquad : t<0 \qquad\qquad : t>0$$

$$\left[c\sqrt{\frac{T}{N\triangle t}} \sum_{j=-N/2}^{N/2-1} d(u+t-j-1/2) \times c^*\sqrt{\frac{T}{N\triangle t}} \sum_{l=-N/2}^{N/2-1} d(u-l-1/2) \right] du$$

where T is the interpulse period and $\triangle t \ll 1$ is the pulsewidth. The energy in the pulse train has been normalized to $|c|^2$. Setting $i=l-j$ and carrying out the integration, we obtain:

$$R_o(t) = |c|^2 \frac{T}{N\triangle t} \sum_{i=-N+1}^{i=N-1} (N - |i|) \, d(t - iT) \qquad (4.10)$$

Equation (4.10) is plotted in figure 4.6.
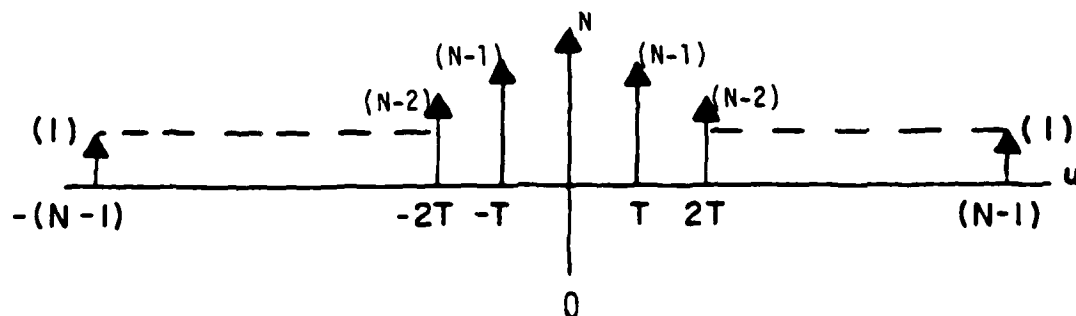
FIGURE 4.6   TIME AVERAGE AUTO-CORRELATION FUNCTION OF FINITE PULSE TRAIN

For the modulated output waveform of the finite pulse train where

$$mT_o \leq T < (m+2)T$$

the resulting average auto-correlation function is

$$R(t) = |c|^2 \frac{T}{N\Delta t} \sum_{i=-m/2}^{i=m/2} (N - |i|)\exp\left\{-\frac{1}{2}\left(\sigma_1^2 + 2\sigma_o^2 \frac{2}{T}\right)|t|\right\} d(t-iT)$$
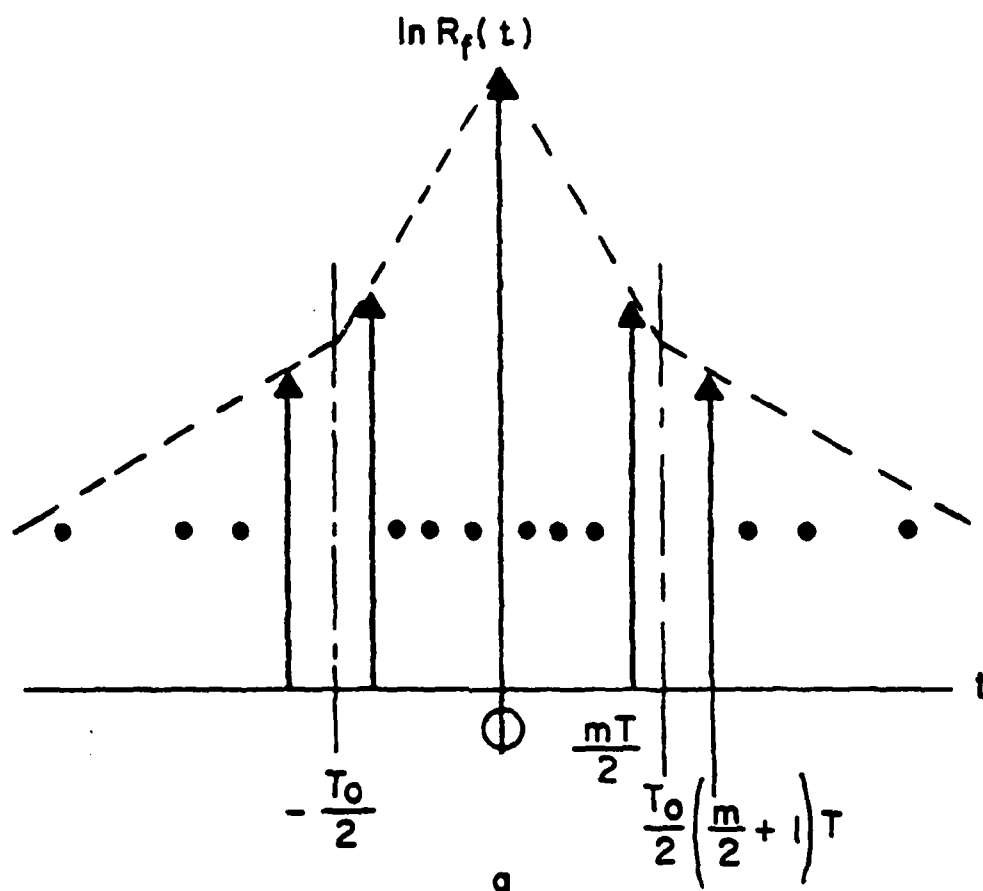
$$: |t| < \frac{T_o}{2}$$

$$R(t) = |c|^2 \frac{T}{N\Delta t} \sum_{\substack{i=-N \\ i \notin [-m/2, m/2]}}^{i=N} (N-|i|)\exp\left\{-\frac{1}{2}\left(\sigma_1^2|t| + 2\sigma_o^2\right)\right\} d(t-iT) \qquad (4.11)$$

$$: |t| > \frac{T_o}{2}$$

Next we determine the average energy spectrum for the finite pulse train. We take the simplified case, $m = 0$. Taking the transform of the above average auto-correlation function (equation (4.11)) we have for $m = 0$, i.e. figure 4.5b,

$$S(w) = |c|^2 \frac{T}{N}\left[N + \exp\{-\sigma_o^2\} \sum_{i=1}^{N-1} (N-|i|)[\exp\{-i(k_1 -jw)T\} + \exp\{-i(k_1 +jw)T\}] \right]$$

$$S(w) = |c|^2 T \left[1 + \exp\{-\sigma_0^2\} \left[\frac{\cos b - e^{-a}}{\cosh a - \cos b} \right.\right.$$

$$\left.\left. - \frac{(1 - e^{-Na}\cos Nb)(\cosh a \cos b - 1) + e^{-Na}\sin Nb(\sinh a \sin b)}{N(\cosh a - \cos b)^2} \right] \right] \tag{4.12}$$

where $a = k_1 T$, $b = w T$, and $k_1 = \dfrac{\sigma_1^2}{2}$

The average energy spectrum for more general correlation functions, i.e. for $m \geq 1$, and where the stationary phase decorrelation time extends beyond the first pulse can also be found in a similar manner. We, however, will not consider this case here because the expressions become quite cumbersome without adding any further insight into the method of analysis presented here.

Since the modulating envelope waveforms, $f_2(u_0)$, modeled for the oscillator were normalized for energy $|c|^2$, the signal to noise ratio in every case would be $|c|^2/N_0$, [4.3] where N_0 is due to thermal additive noise.

The three equations, (4.6), (4.9), and (4.12) provide the spectral characterization for the following instability conditions:

1. <u>Equation (4.6)</u> - Unmodulated CW oscillator signal.

    (4.6a) - "near white" stationary and random walk [4.4]

    (4.6b) - "near white" and intermediate stationary, random walk

2. <u>Equation (4.9)</u> - Infinite pulse train modulated oscillator signal

(4.9a) - "near white" stationary and random walk, stationary
correlation spanning more than one interpulse interval

(4.9b) - stationary white and random walk, stationary correlation
spanning less than one interpulse interval

3. <u>Equation (4.12)</u> - Finite pulse train modulated oscillator signal,
"near white" and random walk, stationary spanning
less than one interpulse interval.

4.2. <u>Random Walk Frequency and Frequency Linear Drift</u>

In this section, we consider the second model, where the phase random process is characterized by random walk frequency and frequency linear drift. First, we treat the oscillator rf signal only. After obtaining expressions for the power spectral density, we will, at the end of this chapter, consider modulation by the finite pulse train.

The auto-correlation function of the oscillator waveform has been determined in equation (2.15). The power spectral density of the oscillator rf signal having a random walk frequency component can be found by evaluating the Fourier transform of equation (2.15). Specifically

$$S_v(w) = \int_{-\infty}^{\infty} \exp\left\{ -\frac{\sigma^2}{6}|t|^3 - \frac{c^2\sigma^2}{2}|t|^2 - jwt \right\} dt \qquad (4.13)$$

To the author's knowledge, no closed form or tabulated solution exists for the above integral. Such solutions, however, do exist for factors of the auto-correlation function appearing in the above integrand [4.5]. These can be used successfully to evaluate the effect that random walk frequency has on power spectral spread. The two factors appearing in the above integral give rise to the random walk frequency along with the offset random frequency. The first factor (first term in the exponential) is solely due to the stationary independent increments. The second factor (second term in the exponential) is due to the long term frequency linear drift, the offset frequency at time, u. (See discussion on equation (C.2) in Appendix C)

4.2.1. Frequency Linear Drift Deconvolution

Equation (2.15), for the auto-correlation function can be written as a product

$$R_b(t) = \exp\left[-\frac{\sigma^2(3c^2|t|^2 + |t|^3)}{6} \right] = \exp\left[-\frac{c^2\sigma^2}{2}|t|^2 \right] \times \exp\left[-\frac{\sigma^2|t|^3}{6} \right]$$

Since the product of the auto-correlation functions implies convolution of the power spectral densities we have

$$S_v(w) = S_2(w) * S(w) \tag{4.14a}$$

where the integral

$$S_2(w) = \int_{-\infty}^{\infty} \exp\left\{-\frac{c_2 \sigma_2^2}{2}|t|^2 - jwt\right\} dt = \frac{1}{\sigma_2}\sqrt{\frac{2\pi}{c_2}}\exp\left\langle -\frac{w^2}{2c_2\sigma_2^2}\right\rangle \tag{4.14b}$$

is the spectrum resulting from the offset frequency (frequency linear drift) at time, u, and

$$S(w) = \int_{-\infty}^{\infty} \exp\left\{-\frac{\sigma_2^2}{6}|t'|^3 - jwt'\right\} dt' \tag{4.14c}$$

is the spectrum resulting from the random walk (stationary independent increments) frequency (referenced to the desired oscillator frequency at u, i.e. t=0). Note [4.6] that equation (4.14b) gives rise to a power spectral density with Gaussian weighting over the spectral variable, w, with mean, zero, and variance, $c_2\sigma_2^2$. This is precisely the assumption we make at time, u, for the offset (long term linear drift) frequency in equation (2.16) and again in equation (C.2b). In those cases, we model the random walk frequency waveform as having a Gaussian offset condition. Clearly, from equation (4.13) and equation (4.14a), we see that the spectrum for the frequency linear drift component given by equation (4.14b) has a dispersive (broadening) effect on the spectrum for the random walk frequency component given in equation (4.14c). The effect which the offset Gaussian distributed frequency has on the final power spectral density arises from the convolution given in

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equation (4.14a).

Time and spectral domain relationships are well known [4.5], [4.6] for the Gaussian model represented by equation (4.14b). In the next section we will consider the evaluation of the stationary independent increment components of the random walk frequency, equation (4.14c).

4.2.2. Series Solution for Related Airy Integral

We will focus attention on the random walk frequency drift effect of equation (4.14c). The integral appearing in that equation is a "related Airy function", properties and applications of which have been considered in the literature [4.5], [4.7]-[4.11]. In order to apply the references in the evaluation of equation (4.14c), we proceed as follows: First we change our variable, t, to

$$t' = -gt \quad : \quad t < 0$$
$$t' = gt \quad : \quad t > 0$$

Equation (4.14c) then becomes

$$S(w) = g \left[\int_{0}^{\infty g^{-1}} \exp\{-\frac{1}{3} t^3 + jgwt\}dt + \int_{0}^{\infty g^{-1}} \exp\{-\frac{1}{3} t^3 - jgwt\}dt \right] \quad (4.15a)$$

where the complex character of g indicates the integration path which will be developed later. Let us write this equation as follows.

$$S(w) = \pi g [Hi(jgw) + Hi(-jgw)] \quad (4.15b)$$

where

$$Hi(x) \underset{=}{\triangle} \frac{1}{\pi} \int_{-\infty}^{\infty} \exp\{ - t^3/3 + xt \} \, dt \quad : \quad x \text{ (real)}$$

and

$$g = (2/\sigma_2)^{1/3} = \left| (2/\sigma_2)^{1/3} \right| \exp(j \frac{2}{3} \pi k) \qquad : k = -1, 0, 1$$

is in general, a complex variable. The complex limits in equation (4.15a) represent the radial path of integration, see figure 4.7, and Hi(w) is the "related Airy function". For t real, equation (4.15) is evaluated by taking the above integral over any one of three paths as shown in figure 4.7. We shall show that the integral results in the same expression regardless of the path of integration, k = -1, 0, 1, taken.
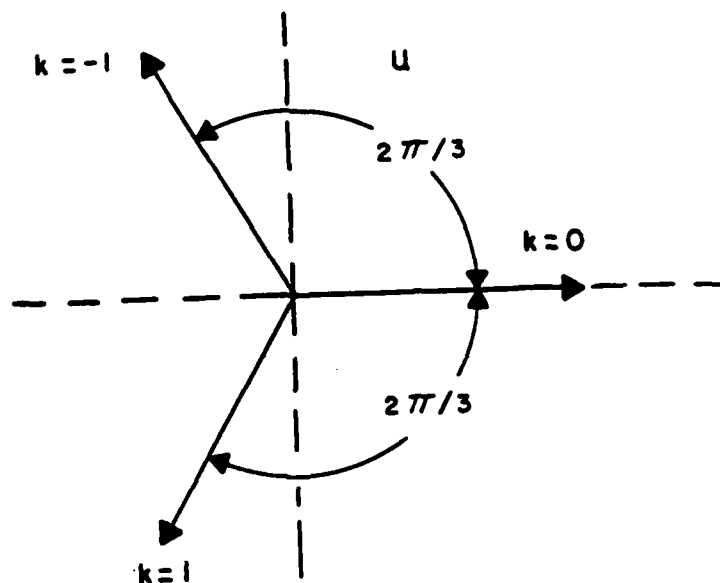
FIGURE 4.7   INTEGRATION PATHS FOR RELATED AIRY INTEGRAL

From equations (6)-(8), (10) and (11) of [4.11], we can evaluate equation (4.15) where k = 0, -1, and +1 correspond to the paths 0 to ∞, 0 to ∞ exp(j2/3 Π), and 0 to ∞ exp(-j2/3 Π) respectively. Let us rewrite equation (4.15) with the following substitution;

$$z = jgw = |w|\exp(j\ \Pi/2)\ |g|\exp(j\ 2k\Pi/3)$$

$$= |z|\exp(j\ [\Pi/2 + 2k\Pi/3]) \qquad : k = 0, -1, 1$$

which preserves the integration paths of figure 4.7. Then equation (4.15) is evaluated using equations (10) and (11) of [4.11], where I_1, I_2, and I_3 in that reference are designated by k = 0, -1, and 1 respectively. We then have

$$S(w) = 3^{-2/3}\ g\ \exp(-j\ \frac{2k\Pi}{3})\ \sum_{r=0}^{\infty} [(z)^r + (-z)^r]\ \exp(-j\ \frac{2k\Pi r}{3})\ \frac{3^{r/3}}{r!}\ (\frac{r}{3} - \frac{2}{3})!$$

Clearly for odd values of r the above equation is zero. Summing over even values of r, multiplying g by the first exponential in the above equation and substituting the above expression for z we have

$$S(w) =$$

$$= 3^{-2/3}\ |g|\ \sum_{r=0}^{\infty} 2|z|^{2r}\ \exp\left[j(\frac{\Pi}{2} + \frac{2k\Pi}{3})2r\right]\ \exp\left[-j\ \frac{2k\Pi}{3} 2r\right]\ \frac{3^{2r/3}}{(2r)!}\ (\frac{2}{3}[r-1])!$$

$$= 2|g|\, 3^{-2/3} \sum_{r=0}^{\infty} |z|^{2r} (-1)^r \frac{3^{2r/3}}{(2r)!} \left(\frac{2}{3}[r-1]\right)! \tag{4.16}$$

which is invariant with k =0, -1, and 1. Furthermore

$$|z| = \left| w \left[\frac{2}{\sigma^2_2} \right]^{1/3} \right|$$

We may therefore take the integral over the path 0 to ∞, i.e. k   with

$$z = jw \left| \frac{2}{\sigma^2_2} \right|^{1/3}$$

in order to determine the spectrum, S(w).

4.2.2.1. <u>Limits of the Series Solution</u>

The series, given in equation (4.16), is useful for small values of w. When using equation (4.16) to compute S(w), it must be kept in mind that the maximum term in the series will be

$$\max\{s\} = (2)^{-2/3} 3^{1/3} \frac{(3^{1/3} p)^{3^{1/3} p}}{(3^{1/3} p)!} \qquad : \ p = |z|$$

Solving this expression for p, we find the index, r, which gives the largest term in the series. This occurs at

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$$r_{(max)} = \frac{3^{1/3}}{2} p_{(max)}$$

It is therefore necessary that equation (4.15) be used only for

$$w < p_{(max)} \left[\frac{\sigma_2^2}{2} \right]^{1/3} = \frac{z(max)}{|g|}$$

in order to be within the precision limits of the computational device.

4.2.3. Asymptotic Expansions for Related Airy Integrals

For cases where

$$w \geq p \left[\frac{\sigma_2^2}{2} \right]^{1/3}$$

asymptotic expansions for equation (4.16) can provide results well within most precision requirements. From Poincare [4.12], Scorer [4.7], and Olver [4.13] the following asymptotic expansion is obtained for sufficiently large negative real x

$$Hi(x) \sim -\frac{1}{\pi x} \left[1 + \frac{1}{x^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3x^3)^s} \right] \quad : \quad x \longrightarrow -\infty \quad (4.17)$$

where the symbol $\sim$ denotes "asymptotic to". The error of the above

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expansion continues to decrease so long as

$$(3s+2)(3s+1)(3s)/s < 3|x|^3$$

$$s < (1/3)|x|^{3/2}$$

for large negative x, say $|x| > 5$. In the complex plane, the asymptotic expansion – with x (real) replaced by z (complex) – holds [4.13] for the analytic continuation Hi(z) of Hi(x), provided that

$$|arg(-z)| \leq \Pi/3 - d,$$

(it holds for $2\ \Pi/3 + d \leq arg(z) \leq 4\ \Pi/3 - d$: it does not hold for $-2\ \Pi/3 + d \leq arg(z) \leq 2\ \Pi/3 - d$: d = an arbitrary positive constant.)

We will next modify the expression in equation (4.17) so that it will be analytically continous along the jw axis. We use a method developed by Olver [4.13] which assures analytical continuity along the imaginary axis through a rotation of the coordinate system.

4.2.3.1. Analytic Continuation at z = ±jw

When evaluating equation (4.17) for

$$z = \pm jgw = gw\ exp(-jB) \quad :B = (\Pi/2)$$

we apply Olver [4.13], theorems 3.2 and 3.3 to the integral, Hi(±jgw), i.e. chapter 11, equation (12.12) [4.13], and use the principal value of

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$$g = \left| \left[\frac{2}{\dfrac{\sigma^2}{2}} \right]^{1/3} \right| \qquad\qquad : k = 0$$

Watson's Lemma, [4.13], states for certain convergence conditions, that the integral

$$I(z) = \int_0^\infty e^{-zt} q(t)\, dt$$

has an asymptotic expansion

$$I(z) \sim \sum_{s=0}^\infty \Gamma \left(\frac{s + \lambda}{u} \right) \frac{a_s}{z^{(s + \lambda)/u}}$$

as $z \longrightarrow \infty$ in the sector $|\arg(z)| < \Pi/2$ where $z^{(s + \lambda)/u}$ has its principal value and $q(t)$ has the Mclaurin expansion as $t \longrightarrow 0+$

$$q(t) \sim \sum_{s=0}^\infty a_s t^{(s + \lambda - u)/u}$$

c.f. theorem 3.2, page 113, [4.13]. For the case of interest the parameters, $\lambda = u = 1$. From theorem 3.3, page 114, [4.13], and the discussion on page 431 [4.13] we may find the analytic continuation of $I(z)$ in the sector, $- 2\,\Pi/3 < \arg(z) < 2\,\Pi/3$ by a rotation of the path of integration by $B = \pm j\,\Pi/6$. Applying theorem 3.2, [4.13], with $t \exp(jB)$ and $z \exp(-jB)$ playing the roles of t and z respectively, we have for the Mclaurin expansion, equation (3 .02), page 113, [4.13],

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$$q(te^{jB}) = \exp(-\frac{1}{3}t^3 e^{j3B})$$

$$= 1 - \frac{2}{3!}e^{j3B}(te^{jB})^3 + \frac{40}{6!}e^{j6B}(te^{jB})^6 + \ldots \quad (4.18a)$$

We see that the constants in the Mclaurin's expansion, page 113, equation (3 .02), [4.13], become

$$a_o = 1, \quad a_3 = \frac{2}{3!}e^{j3B}, \quad a_6 = \frac{40}{6!}e^{j6B}, \quad a_9 = \frac{2240}{9!}e^{j9B}, \quad \text{etc.}$$

The asymptotic expansion, equation (3 .03), page 113, [4.13], becomes

$$I(z) = \frac{1}{z e^{-jB}} - \overline{\left|(4) \frac{2}{3!}e^{j3B}\frac{1}{z^4 e^{-j4B}}\right.}$$

$$+ \overline{\left|(7) \frac{40}{6!}e^{j6B}\frac{1}{z^7 e^{-j7B}}\right.} - \overline{\left|(10) \frac{2240}{9!}e^{j9B}\frac{1}{z^{10} e^{-j10B}}\right.} + \ldots \quad (4.18b)$$

In order to evaluate the integral, I(z) at $z = \pm jw = w \exp(\pm j\Pi/2)$, we rotate the paths of integration by setting $B = \pm(\Pi/6)$ respectively for the $\pm$ operator, i.e.

$$I(\pm jw) = I(jw; B = \Pi/6) \text{ or } I(-jw; B = -\Pi/6)$$

We obtain

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$$I(\pm jw) = \pm \frac{1}{jw} e^{\pm j\Pi/6} + j\frac{2}{w^4} e^{\pm j\Pi/6} \mp \frac{40}{jw^7} e^{\pm j\Pi/6} - j\frac{2240}{w^{10}} e^{\pm j\Pi/6} \pm \ldots$$

$$I(\pm jw) = (\pm 1/jw + 2/w^4 \mp 40/jw^7 - 2240/w^{10} \pm \ldots) \exp(\pm j\Pi/6) \qquad (4.18c)$$

Combining the $\pm$ components as the superposition of the two expressions for $\langle z,B \rangle = \langle +jw, \Pi/6 \rangle$, $\langle -jw, -\Pi/6 \rangle$, we have

$$I(jw) + I(-jw) = (1/w - 40/w^7 + \ldots) + \sqrt{3}(2/w^4 - 2240/w^{10} + \ldots) \qquad (4.18d)$$

Substitution of the parameters from equation (4.15) results in the power spectral density asymptotic expansion.

$$S(w) \sim g \left[\frac{1}{w} - \frac{40}{(gw)^7} + \ldots \right] + \left[\frac{2\sqrt{3}}{(gw)^4} - \frac{2240\sqrt{3}}{(gw)^{10}} + \ldots \right]$$

$$S(w) \sim \frac{1}{w} \left\{ 1 - \frac{1}{3(gw)^6} \sum_{s=0}^{\infty} (-1)^s \frac{(6s+5)!}{(2s+1)! \, 9^s (gw)^{6s}} \right\}$$

$$+ \left\{ \frac{\sqrt{3}}{g^3 w^4} \sum_{s=0}^{\infty} (-1)^s \frac{(6s+2)!}{(2s)! \, 9^s (gw)^{6s}} \right\} \qquad (4.19)$$

where

$$g = \left| (2/\sigma_2^2)^{1/3} \right|$$

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and w is sufficiently large. This is the asymptotic expression for the power spectral density of a cw rf signal having a random walk frequency instability with drift variance σ_2^2.

4.2.4. Effects on Modulating Waveforms

Next we consider the two modulating waveforms, the cw and the finite pulse train.

4.2.4.1. CW Waveform

When the modulating waveform is a cw, with amplitude c, we have for power spectral density, the product of equation (4.19) and $|c|^2$, a constant.

$$S_{cw}(w) = |c|^2 \, S(w) \qquad (4.20)$$

4.2.4.2. Finite Pulse Train

The time average auto-correlation function, Ro(t), of the finite pulse train envelope was previously given on equation (4.10). The auto-correlation function of the rf oscillator signal corrupted by the random walk frequency instability only, i.e. neglecting any offset frequency instability, is found in equation (2.15)

$$R_b(t) = \exp\left\{ -\frac{\sigma_2^2 \, |t|^3}{6} \right\} \qquad (4.21)$$

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Hence, the average auto-correlation function of the modulated output waveform of the finite pulse train is

$$R(t) = R_o(t)R_b(t) = |c|^2 \frac{T}{N\triangle t} \sum_{i=-(N-1)}^{i=(N-1)} (N - |i|) \exp\left\{-\frac{\frac{\sigma^2 |iT|^3}{2}}{6}\right\} d(t-iT)$$

$$(4.22)$$

We determine the average energy spectrum of the random walk frequency rf signal modulated by the finite pulse train by taking the Fourier transform of equation (4.22). The result is

$$S_{fp}(w) = |c|^2 T \left[1 + \sum_{i=1}^{N-1} \frac{N - |i|}{N} \exp\left\{-\frac{\frac{\sigma^2 |iT|^3}{2}}{6}\right\} \cos(wiT) \right] \quad (4.23)$$

A closed form or tabulated solution for equation (4.23) was not attempted due to the analytical difficulties encountered from that expression. A numerical evaluation will be accomplished in following chapter.

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$$I = \int_{x_1}^{x_2} f(x)\, e^{i\phi(x)}\, dx$$

and the tabulation of the function

$$Gi(z) = \frac{1}{\pi} \int_{0}^{\infty} \sin\left(uz + \frac{1}{3} u^3 \right) du$$

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V   NUMERICAL EXAMPLES AND APPROXIMATIONS FOR POWER SPECTRAL DENSITY

In the previous chapter, we have considered the spectral spread due to two groups of phase random processes. Throughout this chapter, we present numerical results obtained for spectral spread. Since the spectral representations are symmetrical, we will present the one sided response only. For some cases, approximations were required and approximate results are presented. As we have done thus far, the two models will be treated separately.

5.1.   White Phase and Random Walk Phase - Oscillator RF Signal with Modulation Waveforms

5.1.1.   CW Waveform

Suppose a CW oscillator has a white phase component with standard deviation

$$\sigma_o = 10^{-10} \text{ radians}$$

over an averaging time of

$$T_o/2 = 10^{-9} \text{ seconds}$$

and a random walk phase component with standard deviation

$$\sigma_1 = 10^{-9} \text{ radians}/\sqrt{\text{sec.}}$$

Then the auto-correlation function of the output waveform is shown in figure 5.1a and the resulting one sided power spectral density obtained from equation (4.6a) is shown in figure 5.1b.

When σ_o, σ_1, and T_o are small, i.e. $\ll 1$, then the power spectral density for the cw oscillator, equation (4.6a), can be approximated over the following frequency intervals:

$$\frac{1}{|c|^2 T_o} S(b) = \frac{1}{k_1} \qquad\qquad : \; b \ll k_1 < k_o \ll \text{II} \quad (5.1a)$$

Definitions for the parameters k_o, k_1 ($k_1 < k_o$), and b, will be found immediately following equation (5.2e).
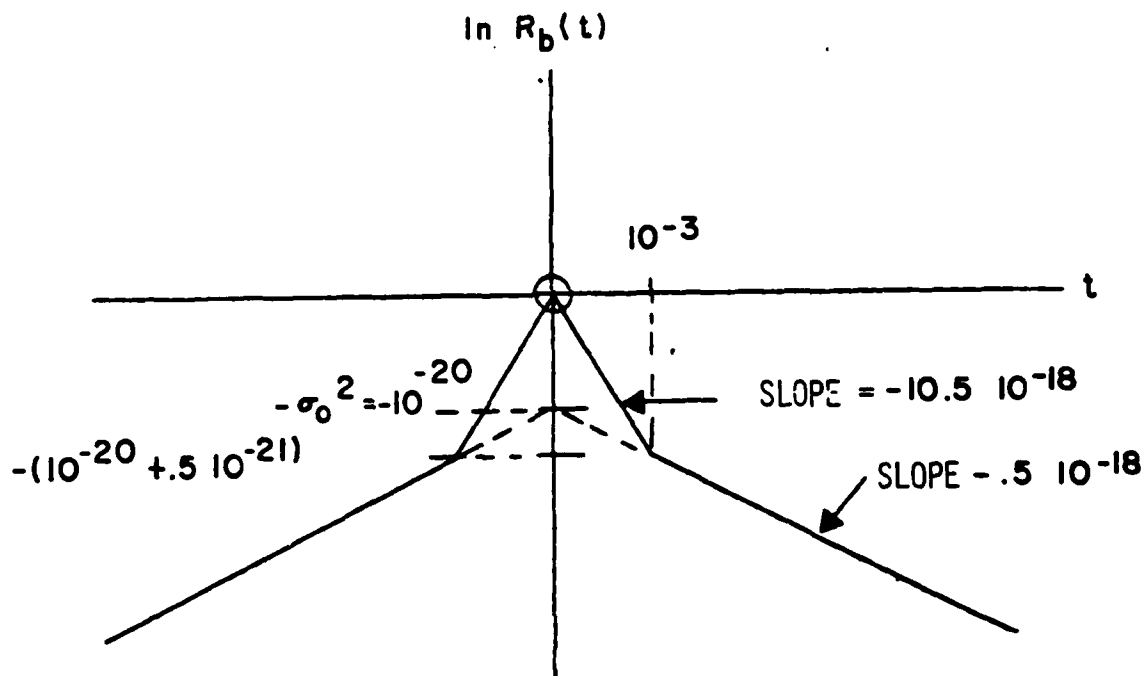
(a) Example – Auto-Correlation Function for
White Phase & Random Walk Phase

FIGURE 5.1   EXAMPLE – CW WAVEFORM

- 108 -
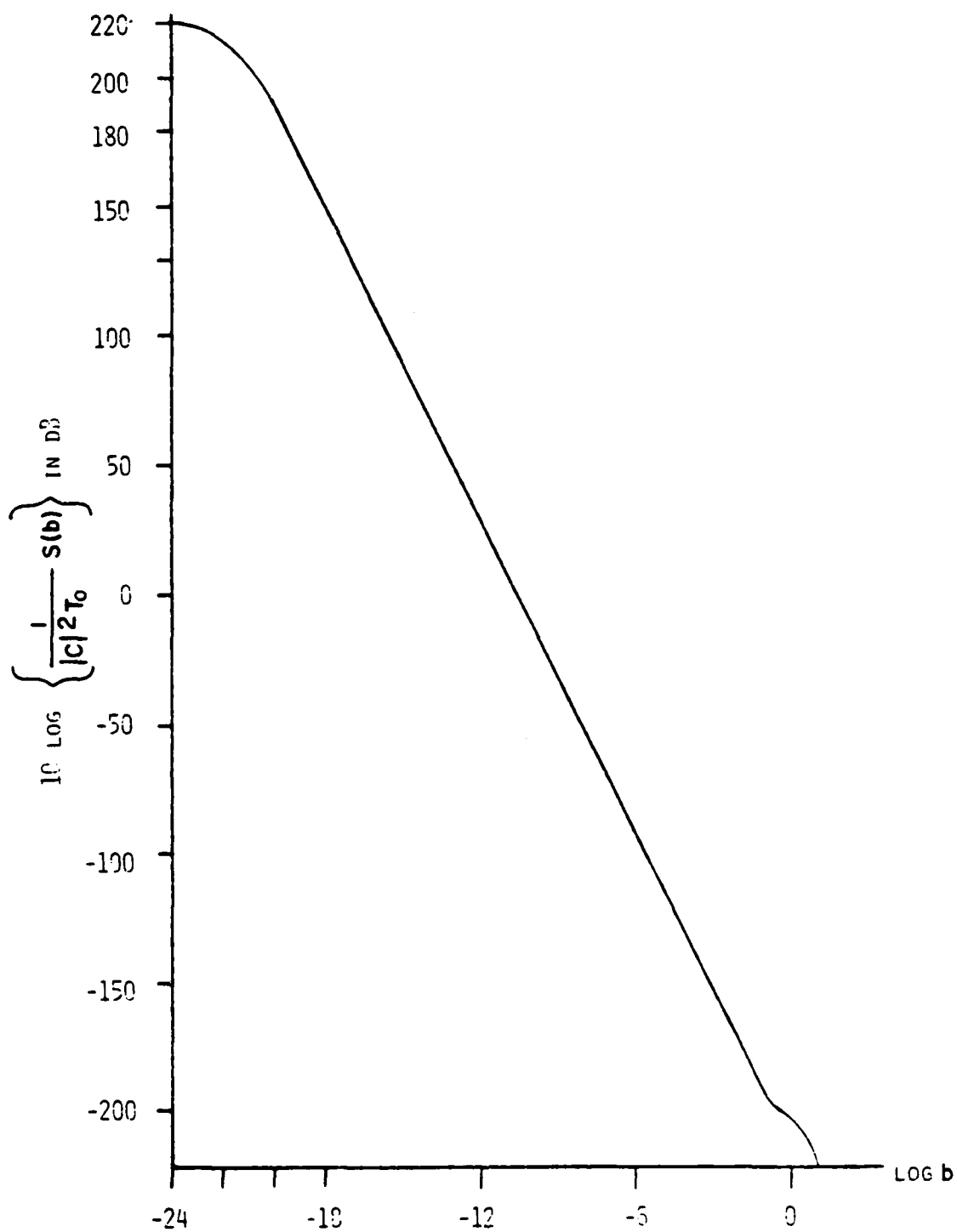
(b) One Sided Power Spectral Density of CW Oscillator

FIGURE 5.1 EXAMPLE - CW WAVEFORM

Equation (5.1a) is obtained through the cancellation of the first two

terms of equation (4.6a), with the approximation of its last term

remaining. Equating b to k_1, the last term of equation (4.6a) becomes

$$= \frac{1}{2k_1} \qquad : b = k_1 < k_0 \ll \Pi \qquad (5.1b)$$

When b is in the range between k_1 and k_0, still very small, the trigono-

metric functions in the numerators of equation (4.6a) are approximated

by one and zero for the cosine and sine respectively. The last becomes

$$= \frac{k_1}{k_1^2 + b^2} \qquad \begin{array}{l} : k_1 < b < k_0 \\[1em] : k_0 < b \ll \Pi/2 \end{array} \qquad (5.1c)$$

Equation (5.1d) is obtained as follows: With k_1 and k_0 very small

the bracketed part of equation (4.6a) is written

$$\frac{1}{b^2} \left[k_0 - (1-k_0) [k_0 \cos b - b \sin b - k_1 \cos b + b \sin b] \right]$$

setting $1-k_0 \simeq 1$, cancelling terms and factoring the result, we have

$$= \frac{k_0 - (k_0 - k_1) \cos b}{b^2} \qquad : k_0 \ll b < \Pi/2 \qquad (5.1d)$$

When b is equal to $\Pi/2$ it dominates all other terms in equation (4.6a).

The cosine terms are zero, the sine terms are one, and we have

$$\frac{1}{b^2} (k_0 - b + b) \qquad \text{which leaves us with}$$

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$$= \frac{k_o}{b^2} \qquad\qquad : \; b = \Pi/2 \qquad (5.1e)$$

In the above expressions we have made the following substitutions.

$$k_o = \frac{\sigma_1^2 T_o}{4} + \sigma_o^2, \quad k_1 = \frac{\sigma_1^2 T_o}{4} < k_o, \quad \text{and} \quad b = w \frac{T_o}{2}.$$

Note that other notations [5.1], [5.2] can be related to the above parameters

$$k_o = h_o \frac{T_o}{4} + \frac{h_2}{(2\,\Pi)^2}, \qquad k_1 = h_o \frac{T_o}{4}$$

Figure 5.2a shows the relationship of the parameters in equation (5.1) to the auto-correlation function for the cw oscillator signal. Figure 5.2b illustrates the plotting of the power spectral density curve.

5.1.2. Infinite Pulse Train

We can approximate equation (4.9b) and obtain the powel spectral density when the stationary component is white. Approximations to the infinite pulse train example are made as follows:

$$\ln R_f = \ln |C|^2 + \ln R_b(t)$$

$$K_0 = \frac{\sigma_1^2 T_0}{4} + \sigma_0^2$$

$$K_1 = \frac{\sigma_1^2 T_0}{4}$$

$$\beta = \omega \frac{T_0}{2}$$

$T_0/2$

$\ln |C|^2$

$$\text{SLOPE} = -K_0 \frac{2}{T_0}$$

σ_0^2

$$\text{SLOPE} = -K_1 \frac{2}{T_0}$$

K_1
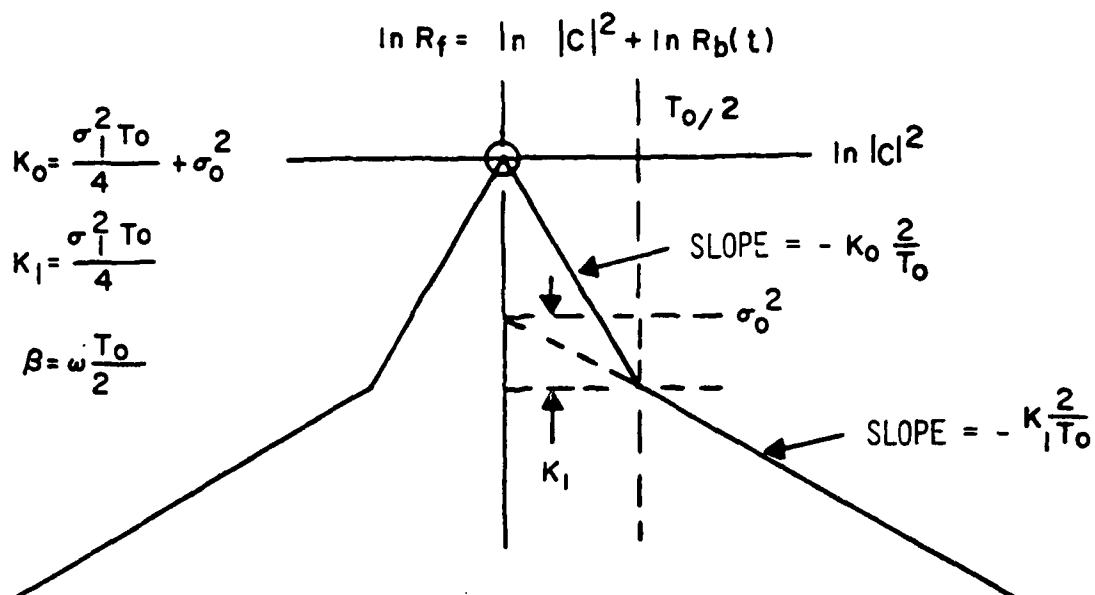
(a) Correlation Function Parameters

(b) OSCILLATOR POWER SPECTRAL DENSITY
NEAR WHITE PHASE & RANDOM WALK PHASE

$\frac{1}{K_1}$

$\frac{1}{2K_1}$

$$\frac{K_1}{K_1^2 + b^2}$$

$$\frac{K_1}{b^2}$$

$$\frac{K_0(1-\cos b) + K_1 \cos b}{b^2}$$

$$\frac{K_0}{b^2}$$

$$\ln\left\{\frac{S(b)}{|C|^2 T_0}\right\}$$

O K_1 $\pi/2$ b
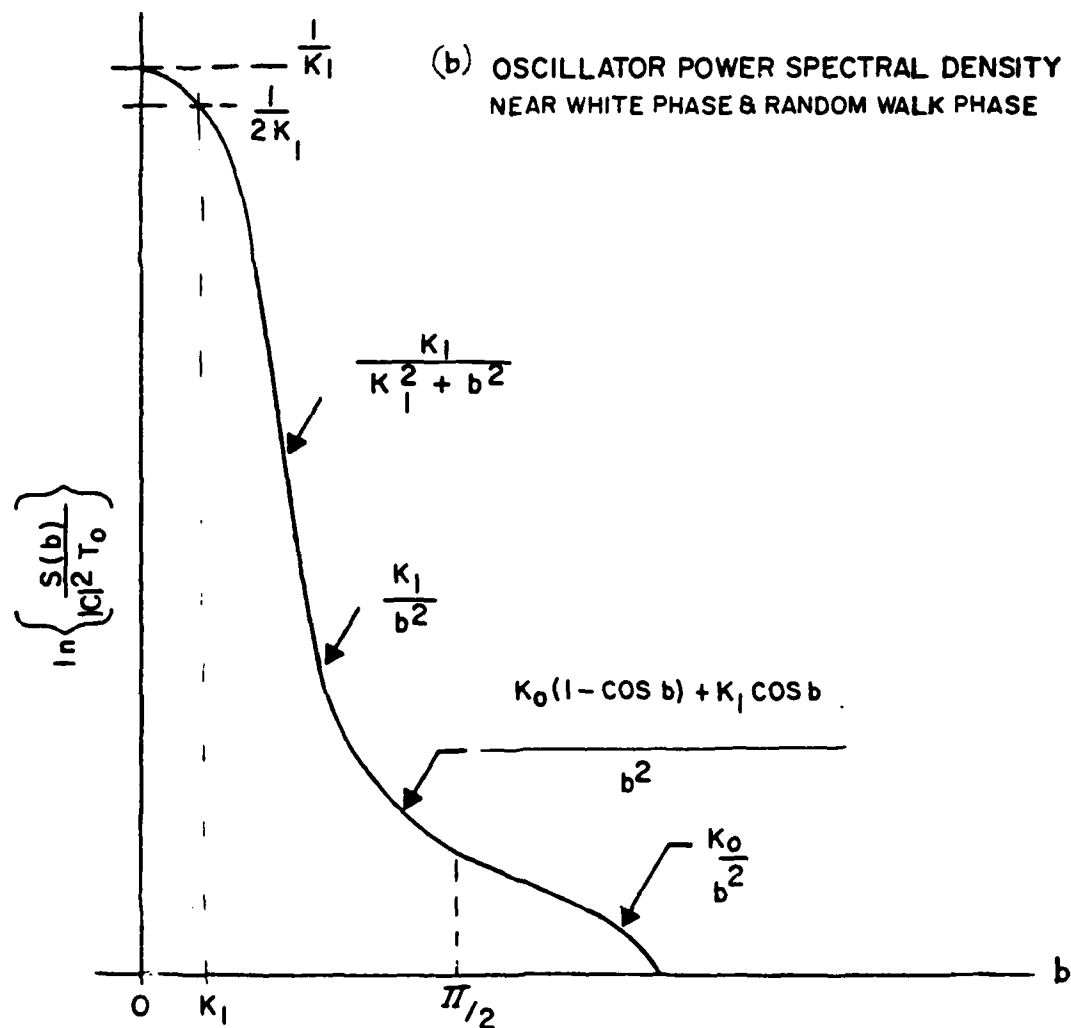
FIGURE 5.2   DEPENDENCE OF AUTO-CORRELATION FUNCTION AND POWER SPECTRAL
DENSITY ON PHASE STATISTICS - CW WAVEFORM

Given, the average power spectrum for the infinite pulse train, equation (4.9b), and denoting its left side as a normalized term we have

$$S_N(w) = \frac{S(w)}{|c|^2 T} = [\; 1 + \exp(-\sigma_o) \;\frac{2 \cos b - \exp(-a)}{\cosh a - \cos b} \;],$$

where $a \ll 1$ and $\sigma_o^2 \ll 1$. For the following conditions: $S_N(w)$ is

$b = 0$:

$$\tilde{}_o [\; 1 + (1 - \sigma^2) \;\frac{1 - (1 - a)}{1 + a^2/2 - 1} \;] \tilde{}_o 1 + (1 - \sigma^2)_o \frac{2}{a} \tilde{} \frac{2}{a}$$

$b = a$:

$$\tilde{}_o [\; 1 + (1 - \sigma^2) \;\frac{\cos a - \exp(-a)}{\cosh a - \cos a} \;]$$

$$\tilde{}_o [\; 1 + (1 - \sigma^2) \;\frac{1 - a^2/2 - (1 - a + a^2/2)}{1 + a^2/2 - 1 + a^2/2} \;]$$

$$\tilde{} 1 + (1 - \sigma^2)_o \frac{a - a^2}{a^2} \tilde{} 1 + (1/a - \sigma^2_o \Big/ a - 1 + \sigma^2)_o \tilde{} 1/a$$

$a \ll b \le \Pi$:

$$\tilde{}_o 1 + \exp(-\sigma^2) \;\frac{\cos b - \exp(-a)}{\cosh a - \cos b}$$

$$\tilde{}_o 1 + (1 - \sigma^2) \;\frac{\cos b - (1 - a)}{1 - \cos b}$$

$$\cong 1 + (1 - \sigma_o^2)\left(\frac{a}{1 - \cos b} - 1\right)$$

$$\cong 1 + \frac{a}{1 - \cos b} - 1 - \frac{a\,\sigma_o^2}{1 - \cos b} + \sigma_o^2 \cong \sigma_o^2 + \frac{a}{1 - \cos b},$$

The approximations are thus summarized as follows:

$$\frac{1}{|c|^2 T}\, S(b) = \frac{2}{a} \qquad\qquad\qquad : \ b = 0$$

$$= \frac{1}{a} \qquad\qquad\qquad : \ b = a$$

$$= \sigma_o^2 + \frac{a}{1 - \cos b} \qquad\qquad : \ a \ll b \le \Pi \qquad (5.2)$$

where $a = k_1 T$, $k_1 = \dfrac{\sigma_1^2}{2}$, and $b = wT$.

Equation (5.2) is sketched for $0 \le b \le \Pi$ in figure 5.3. Note that the peak of the response is limited by the random walk component through the parameter, a. The noise floor (null depth) is determined by the white phase component, the random walk phase component, and frequency.
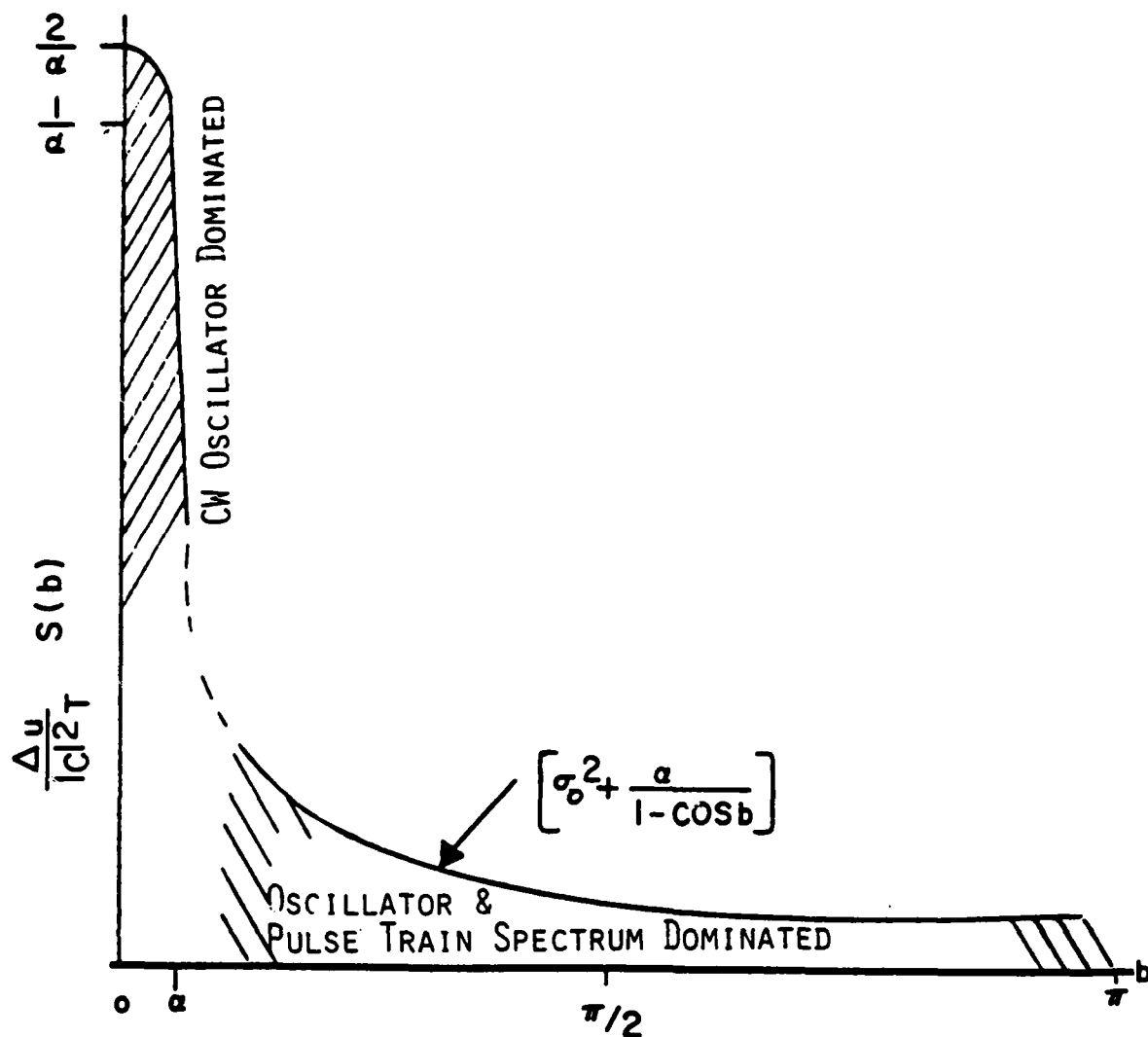
FIGURE 5.3   ONE SIDED AVERAGE ENERGY SPECTRUM - INFINITE PULSE TRAIN

5.1.3.   Finite Pulse Train

Suppose a train of N=32 pulses with interpulse period T=.001
modulates an oscillator.   Let the oscillator have a white phase
component with standard deviation

$$\sigma_o = 10^{-8}$$

over an averaging time less than one interpulse period, T, and a random
walk phase component with standard deviation

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$$\sigma_1 = 10^{-7} \text{ radians/} \sqrt{\text{sec}}$$

The pulse train is shown in figure 5.4a, the auto-correlation function of the output waveform is shown in figure 5.4b, and the resulting average energy spectrum obtained from equation (4.12) is shown in figure 5.4c. Table 5.1 gives the peaks and nulls of the average energy spectrum from b=0 to b=Π. In plotting the spectrum of the finite pulse train, we observe that a number of approximations to equation (4.12) can simplify the computations. These simplifications allow us to determine the peaks of the spectrum sidelobes and the null depths, and are written below:

For $0 < \sigma_o$, $\sigma_1 \ll 1$, $a = \dfrac{\sigma_1^2 T}{2}$, $b = wT$ we have for the

Mainlobe Peak $\quad \dfrac{1}{|c|^2 T} S(b) = N \qquad\qquad : b = n\,\Pi$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad : 0 \leq n$ even

Lowest Peak $\quad \dfrac{1}{|c|^2 T} S(b) = \sigma_o^2 + a \qquad : b = n\,\Pi$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad : n$ odd

Sidelobe Peaks $\quad = \dfrac{2}{N(1-\cos b)} \qquad\qquad : b = n\,\Pi/N$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad : 0 < n < N; \; n$ odd

Null Depth $\quad = \sigma_o^2 + \dfrac{2a}{1-\cos b} \qquad : b = n\,\Pi/N \quad (5.3)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad : 2 \leq n < N; \; n$ even

$$C\sqrt{\dfrac{T}{N\Delta t}}$$

$\longleftarrow$ T $\longrightarrow$

31 T

(a) N = 32 FINITE PULSE TRAIN

CORRELATION FUNCTION

N = 32

$T = 10^{-3} = 1$ MS

$\sigma_0 = 10^{-8}$

$\sigma_1 = 10^{-7}$

$1\left\{\dfrac{\Delta u}{|c|^2 T} R_f(iT)\right\}$

EXP (-10^{-16})

$\dfrac{30}{32}$ EXP$(-10^{-17}-10^{-16})$

$\dfrac{31}{32}$EXP$(-1/2\ 10^{-17}-10^{-16})$

iT

-3   -2   -1   0   1   2   3

INCREMENTAL TIME SHIFT

(b) CORRELATION FUNCTION OF PRODUCT OF FINITE PULSE TRAIN,
NEAR WHITE PHASE (M=0), AND RANDOM WALK PHASE

FIGURE 5.4   PROPERTIES OF MODULATED 32 PULSE TRAIN

$$N = 32$$
$$T = 10^{-3}$$
$$\sigma_0 = 10^{-8}$$
$$\sigma_1 = 10^{-7}$$

NUMBER OF PULSES $N = 32$
INTERPULSE PERIOD $T = 10^{-3}$
WHITE STANDARD DEVIATION $\sigma_0 = 10^{-8}$
RANDOM WALK STANDARD DEV $\sigma_1 = 10^{-7}$

$$a = 5 \cdot 10^{-18}$$
$$\sigma_0^2 = 10^{-16}$$

Vertical axis: $\dfrac{\Delta t}{|c|^2 T} S(\omega)$ SCALE: 1   SCALE: 10^{-16}

$32\ b/\pi$
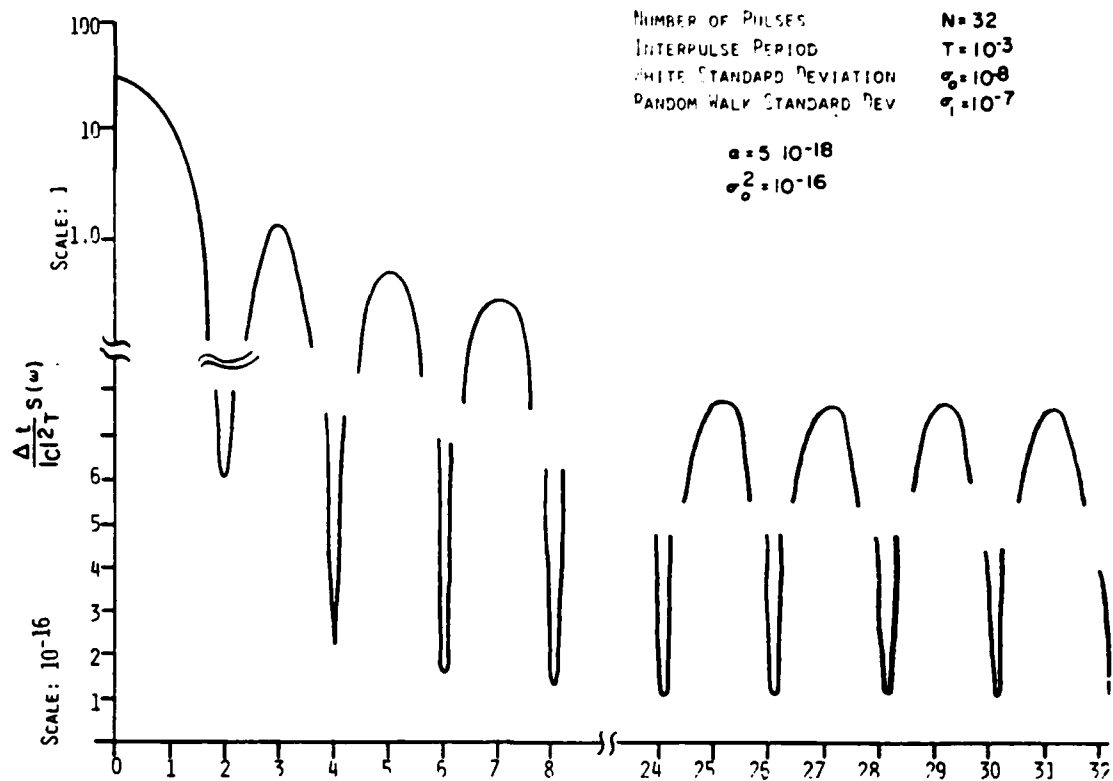
(c)   One Sided Average Energy Spectrum – White & Random Walk Phase

FIGURE 5.4   PROPERTIES OF MODULATED 32 PULSE TRAIN

TABLE 5.1   SPECTRUM PEAKS AND NULLS – FINITE PULSE TRAIN

$N=32$, $T=10^{-3}$, $\sigma_0=10^{-8}$, $\sigma_1=10^{-7}$			
$\dfrac{32}{\pi}\beta$	$\dfrac{\Delta t}{|c|^2 T} S(\omega)$	$\dfrac{32}{\pi}\beta$	$\dfrac{\Delta t}{|c|^2 T} S(\omega)$
0	32	16	$1.100 \; 10^{-16}$
1	12.98	17	.057
2	$6.20 \; 10^{-16}$	18	$1.084 \; 10^{-16}$
3	1.45	19	.048
4	$2.31 \; 10^{-16}$	20	$1.072 \; 10^{-16}$
5	.53	21	.042
6	$1.59 \; 10^{-16}$	22	$1.064 \; 10^{-16}$
7	.28	23	.038
8	$1.34 \; 10^{-16}$	24	$1.059 \; 10^{-16}$
9	.17	25	.035
10	$1.23 \; 10^{-16}$	26	$1.055 \; 10^{-16}$
11	.118	27	.033
12	$1.16 \; 10^{-16}$	28	$1.052 \; 10^{-16}$
13	.008	29	.032
14	$1.12 \; 10^{-16}$	30	$1.050 \; 10^{-16}$
15	.069	31	.031
		32	$1.050 \; 10^{-16}$

The spectrum sidelobe peaks and null depths (upper and lower spectrum limits) are plotted in figure 5.5 for various parameters as indicated. These plots are for uniform weighted pulse trains. With some effort it is a straight forward procedure to obtain the spectrum for a weighted train of pulses. Note from equation (5.3) and figure 5.5 that for the finite pulse train the peak response is determined by the number of pulses and frequency, w; and the null depths are determined by the white phase component σ_o^2, the random walk phase component, σ_1^2, and the frequency, w for all but large values of the phase variance. When the variance is large, then the spectral spread deteriorates beyond that of the sidelobe and mainlobe peaks as shown in figure 5.5b.
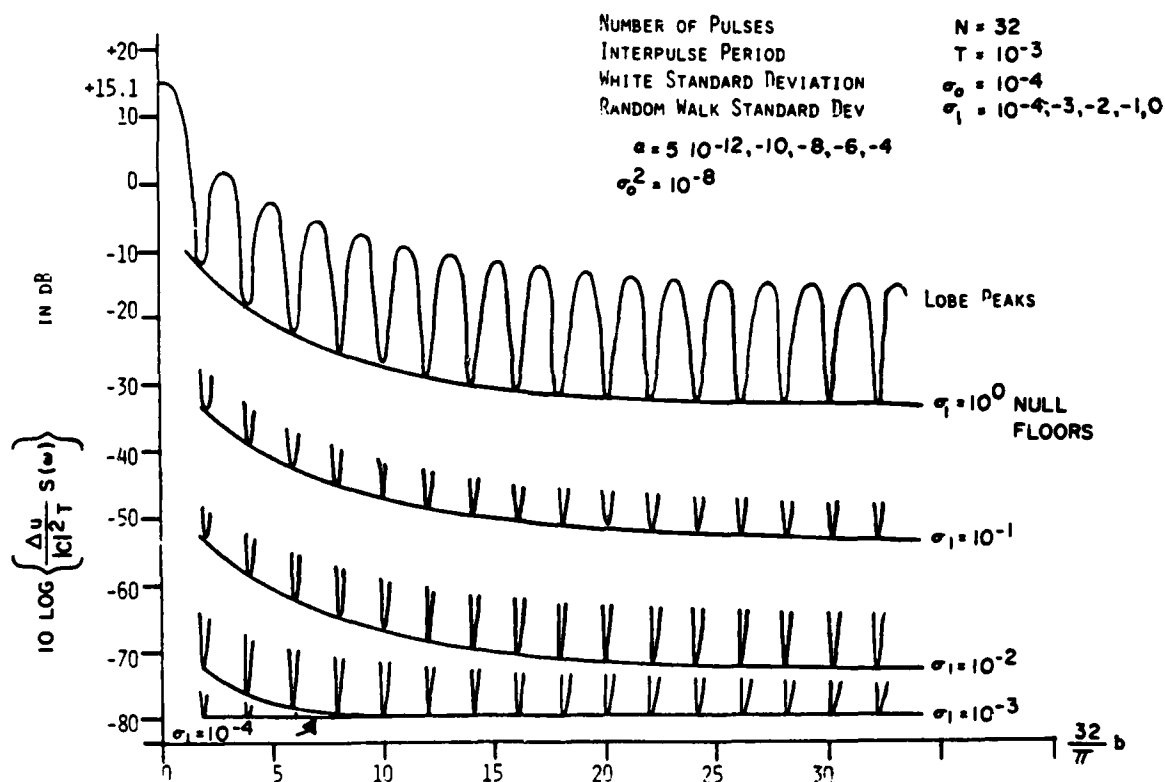
NUMBER OF PULSES $\qquad$ $N = 32$
INTERPULSE PERIOD $\qquad$ $T = 10^{-3}$
WHITE STANDARD DEVIATION $\qquad$ $\sigma_o = 10^{-4}$
RANDOM WALK STANDARD DEV $\qquad$ $\sigma_1 = 10^{-4}, -3, -2, -1, 0$

$a = 5 \; 10^{-12}, -10, -8, -6, -4$

$\sigma_o^2 = 10^{-8}$

(a)   Response - Average Energy Spectrum

FIGURE 5.5   SPECTRUM - WHITE PHASE & RANDOM WALK PHASE - 32 PULSE TRAIN

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NUMBER OF PULSES N=32
INTERPULSE PERIOD T=10^{-3}
WHITE STANDARD DEVIATION $\sigma = 10^{-3}$
RANDOM WALK STANDARD DEVIATION $\sigma_1 = 10^0, 10^1, 3 \cdot 10^1$
$a = 5 \cdot 10^{-4}, -2, -1$
$\sigma_0^2 = 10^{-8}$
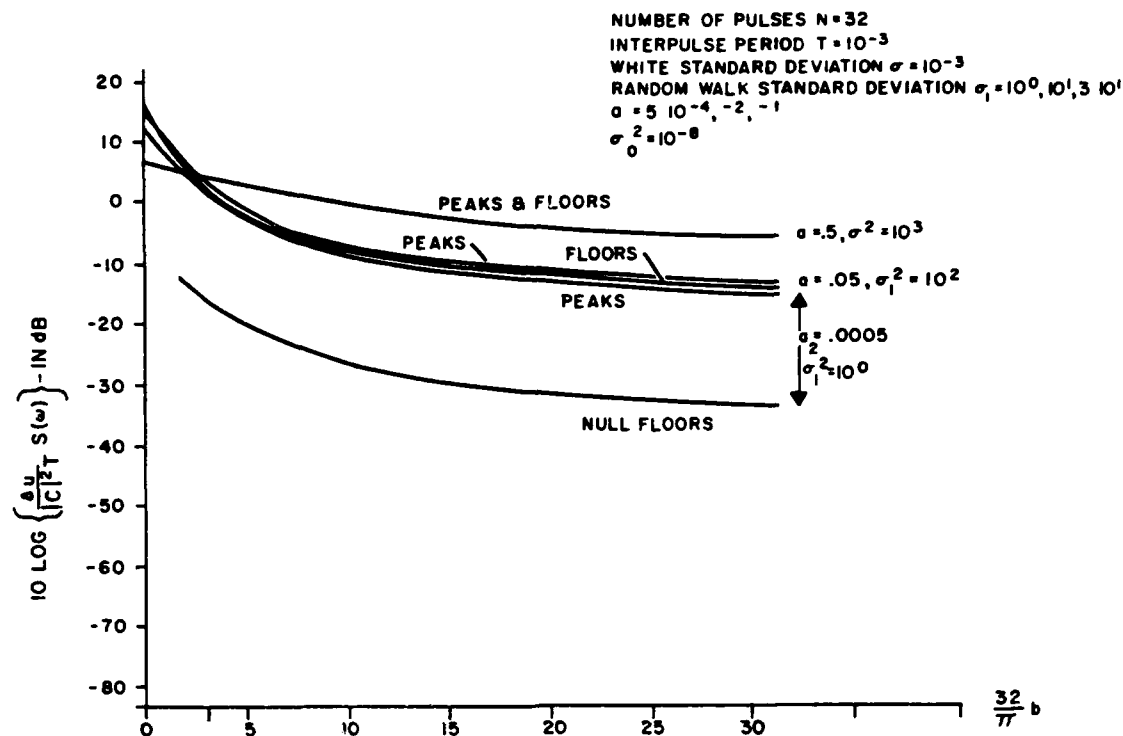
(b) Sidelobe Peaks & Null Depths

NUMBER OF PULSES   N = 32
INTERPULSE PERIOD   T = 10^{-3}
WHITE STANDARD DEVIATION σ_0 = 10^{-6}
RANDOM WALK STANDARD DEV σ_1 = $10^{-6,-5,-4,-3,-2}$
$a = 5 \cdot 10^{-16,-14,-12,-10,-8}$
$\sigma_0^2 = 10^{-12}$
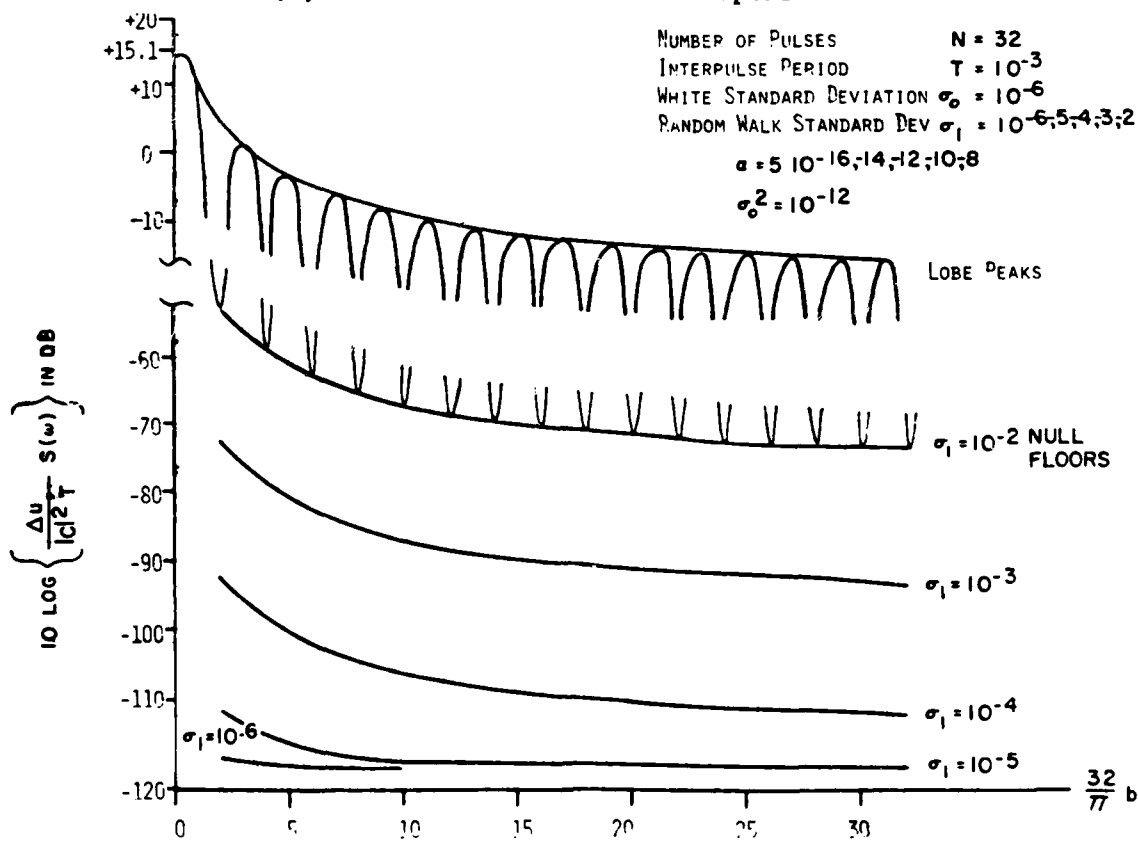
(c)   Response – Average Energy Spectrum

FIGURE 5.5   SPECTRUM – WHITE PHASE & RANDOM WALK PHASE – 32 PULSE TRAIN

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5.2. Random Walk Frequency – Oscillator RF Signal with Modulating Waveforms

To evaluate the effects of spectral spreading of an oscillator when random walk frequency instability is present, we use the exponential expansion, equation (4.16), and the asymptotic expansion, equation (4.19), in normalized form. We perform the normalization by setting the argument, $gw = ci$, where

$$g = \left| \left[\frac{2}{\frac{2}{\sigma_2}} \right]^{1/3} \right|$$

We obtain

$$\frac{1}{g} s_v(w) = 2(3)^{-2/3} \sum_{r=0}^{\infty} \frac{\left| 3^{1/3} ci \right|^{2r}}{(2r)!} (-1)^r \left(\frac{2}{3}[r-1] \right)! \qquad (5.4a)$$

for the exponential expansion and

$$\frac{1}{g} s_v(w) = \frac{1}{ci} \left[1 + \sum_{s=0}^{\infty} (-1)^s \frac{(6s+2)! \, [\sqrt{3}\,(ci)^3 - (6s+4)(6s+5)]}{(2s)! \, 9^s \, (ci)^{6s+6}} \right] \qquad (5.4b)$$

for the asymptotic expansion. Figure 5.6 illustrates the behavior of the oscillator power spectral density with normalized frequency, i.

For the finite pulse train, the average energy spectrum is found through a numerical evaluation of the Fourier Transform and is shown in figure 5.7.

$$\frac{1}{g} S_v(\omega) = 2 \times 3^{-2/3} \sum_{r=0}^{\infty} \frac{\left|3^{1/3} c_i\right|^{2r}}{(2r)!} (-1)^r \left[\frac{2}{3}(r-1)\right]!$$

EXPONENTIAL EXPANSION

ASYMPTOTIC EXPANSION

$$\frac{1}{g} S_v(\omega) = \frac{1}{c_i}\left[1 + \sum_{S=0}^{\infty} (-1)^S \frac{(6S+2)! \left[\sqrt{3}(c_i) - (6S+4)(6S+5)\right]^3}{(2S)! \, 9^S (c_i)^{6S+6}}\right]$$

NORMALIZED FREQUENCY, LOG c_i

POWER SPECTRAL DENSITY OF OSCILLATOR
AND RANDOM WALK FREQUENCY INSTABILITY
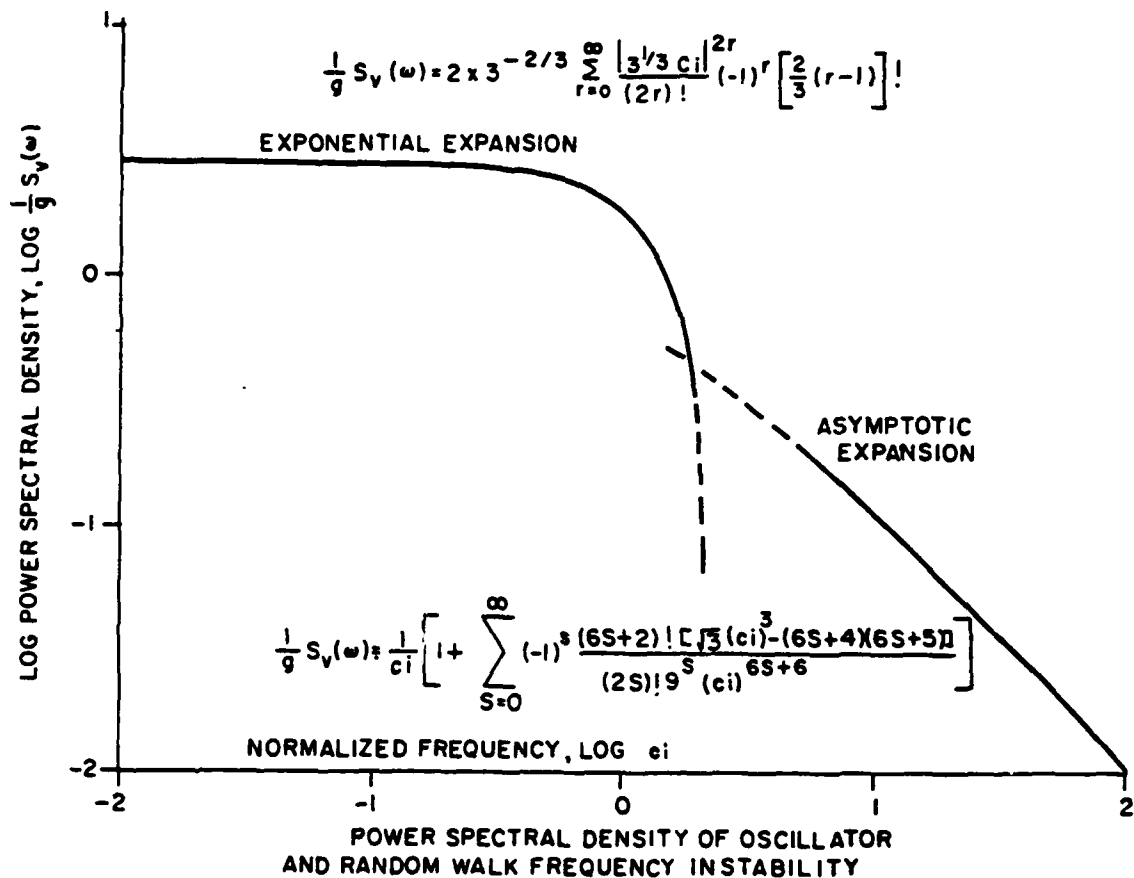
FIGURE 5.6   POWER SPECTRAL DENSITY – RANDOM WALK FREQUENCY INSTABILITY

$$\frac{\Delta T}{|C|^2 T} S(\omega) = \left[1 + 2 \sum_{i=1}^{N-1} \frac{N-i}{N} \exp\left(-\frac{1}{3} K i^3\right) \cos\left(\pi \frac{k}{N} i\right)\right]$$

$$\pi \frac{k}{N} = \omega T \quad K = \left(\frac{\sigma_y^2}{2} T^3\right)$$

FOR $K = 5 \times 10^{-20}, 5 \times 10^{-10}, 5 \times 10^{-7}, 5 \times 10^{-4}, 5 \times 10^{-3}$

N=32
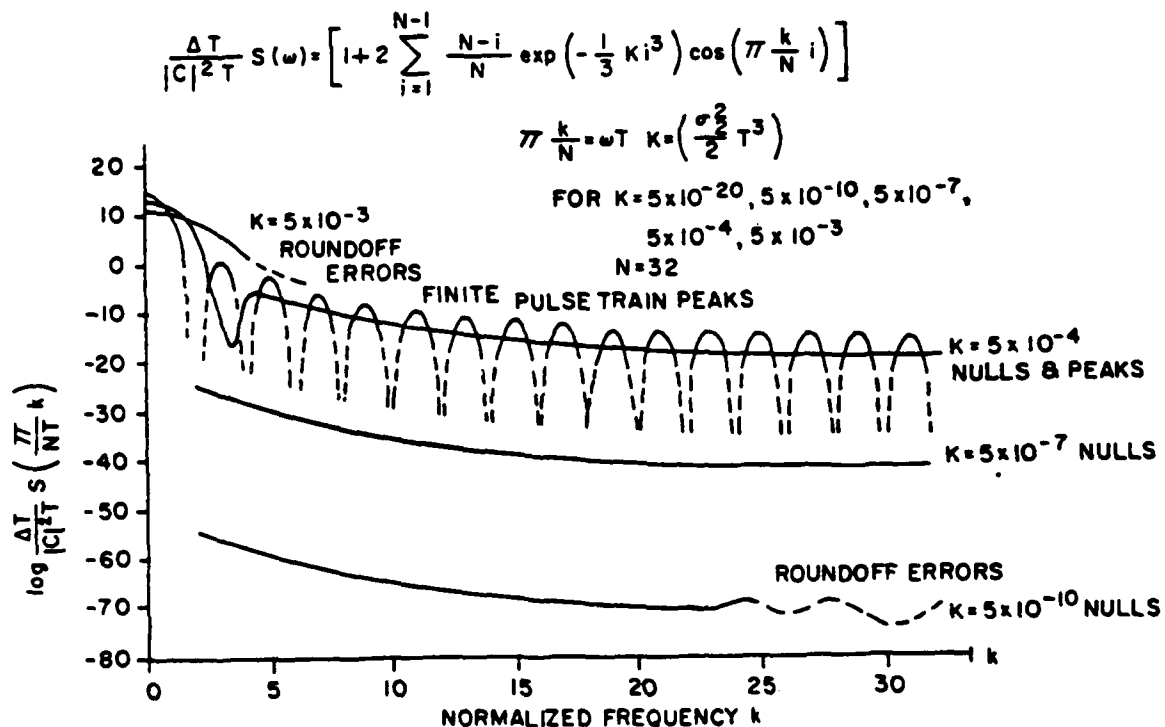
FIGURE 5.7   AVERAGE ENERGY SPECTRUM – RANDOM WALK FREQUENCY &
32 FINITE PULSE TRAIN

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We normalize equation (4.23) by setting

$$wT = \Pi k/N \quad \text{and} \quad K = T^3 \sigma_2^2 / 2$$

Hence

$$\frac{1}{|c|^2 T} s(k) = \left[1 + 2 \sum_{i=1}^{N-1} \frac{N-i}{N} \exp(-\frac{1}{3} Ki) \cos(\Pi \frac{k}{N} i) \right] \quad (5.5)$$

For K=0, the average energy spectrum density is that of a perfectly stable pulse train. Table 5.2 shows the percent deterioration of the average energy spectrum for various values of the parameter, K.

The parameter used by a number of investigators [5.1], [5.2], [5.3]

$$h_{-2} = \sigma_2^2$$

corresponds to the frequency drift variance, σ_2^2, adopted here.

TABLE 5.2

Δ% DETERIORATION OF POWER SPECTRAL DENSITY WITH PARAMETER, K

K	MAINLOBE	NEAR SIDELOBES	FAR SIDELOBES
$5 \ 10^{-10}$	$> -10^{-4}$	$> -10^{-5}$	$> -10^{-5}$
$5 \ 10^{-7}$	-0.054	-0.26	-0.29
$5 \ 10^{-4}$	-27.67	-54.2	-49.95
$5 \ 10^{-3}$	-59.18	ROUND OFF ERRORS	ROUND OFF ERRORS

REFERENCES

[5.1]   Barnes, J. A., et al, "Characterization of Frequency Stability",
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[5.3]   Howe, D. A., "Frequency Domain Stability Measurements: A Tutorial
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VI SUMMARY AND CONCLUSIONS

6.1. Summary

We have developed a procedure for finding the degradation in spectral resolution caused by the phase instability of the system coherent oscillator. Phase instability was modeled as a superposition of a variety of random processes including a "near white" Gaussian process, a non-stationary phase and frequency random walk (Wiener) process, instabilities at some reference time, and long term drift effects. The procedure involved determining the covariance matrix of the phase process, obtaining the auto-correlation function and transforming to the power spectral density. Modulated signals, i.e., signals obtained by modulating the oscillator output signal by some deterministic waveform were also included. Three specific waveforms were treated: cw, infinite pulse train and finite pulse train. Closed form expressions were derived. Approximations were used for low level phase instability. In our work, the pulse width of the single pulse is assumed to be much smaller than the interpulse period so that its effect may be neglected. The results obtained here may be applied to communication and radar systems.

In addition, stationarity and ergodicity properties of the oscillator rf signal containing the instability properties were examined and results were presented in terms of a number of theorems.

6.2. Conclusions

The following conclusions are drawn from our work.

1) The auto-correlation function of the oscillator rf signal is determined by evaluating the characteristic function of the phase random process at appropriate values of the argument.

2) For random walk frequency, the power spectral density of the rf signal is determined through evaluating a "related Airy integral" in the appropriate region of the complex plane. For typical values of the drift frequency variance, i.e. very small, asymptotic expansions are useful for even nominal values of frequency.

3) The rf signal as effected by random walk frequency instability is not ergodic.

4) The expressions for power spectral density derived herein establish the lower end of the system dynamic range.

   o The variance of the white phase component establishes a lower limit for the null depths of the power spectral density of the infinite pulse trains considered. For cw oscillators, this variance in conjunction with the random walk variance establishes the spurious spectral level for large frequency values, w.

o For small values of frequency, w, the variance of the random walk phase component establishes the peak value and the roll-off rate for the power spectral density of the infinite pulse train and the cw oscillator. This variance in conjunction with the variance of the white phase component established the upper limit for the spurious spectral level of the infinite and finite pulse train.

o For small levels of instability, the number of pulses in the finite pulse train determines the peak main response and peak side lobe response of the power spectral density. When the instabilities become very large, i.e. the variances increase without bound, then the side lobe level increases with a complete deterioration of the pattern including the main lobe.

6.3. Future Work

The expressions generated herein have treated white and random walk phase separately from random walk frequency. Hence, the assessment of oscillators when subjected to internal or external white noise sources on one end or environmental conditions on the other end can be achieved. Expressions representing the composite case; white phase, random walk phase, and random walk frequency is a worthwhile extension of these results and deserves further work.

Computer simulations of these phase random processes would verify the results and lead to an experimental approach for evaluating oscillators in the laboratory.

Modeling of flicker noise should be considered from the point of view of physical phenomenon as was done with the phase instability models introduced here.

An investigation into the synchronization and timing of multiple interconnecting systems should be performed using the oscillator instability models developed in this dissertation.

The work performed in this dissertation can, with some modifications, be applied to the modeling of random effects in antenna apertures. The analogies between time-frequency and aperture-pattern can then be used to evaluate the limitations which random effects have on the performance of antenna systems.

APPENDIX A

A.0. Characteristic Functions for Determining Moments

of the RF Signal, b(u).

The characteristic function can provide any order moment [A.1] of
the signal, b(u), for any distribution of the phase random vector, $\underline{z}$.
For example, the second and fourth order moment required often in the
text will now be evaluated for the case when the random phase vector, $\underline{z}$,
is joint Gaussian distributed. From Davenport and Root [A.2], pp 54 and
152, equation (8-44), the Gaussian joint characteristic function is $F_{\underline{z}}(\underline{p})$

$$F_{\underline{z}}(jp_1, \cdots, jp_N) \triangleq E[\exp(j \sum_{n=1}^{N} p_n z_n)] = E[\exp(j\underline{p}^T \underline{z})]$$

$$= \exp(-1/2\ \underline{p}^T M\underline{p}) \tag{A.1}$$

where $\underline{z}$ is a joint Gaussian random vector

$$\underline{z} \triangleq \begin{bmatrix} z(u+s+t) \\ z(u+t) \\ z(u+s) \\ z(u) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

with covariance matrix

$$M_{\underline{z}} = E[\ \underline{z}\,\underline{z}^T\ ] = [\ m_{ij}\ ] \qquad : 1, \; \cdot \; \cdot \; 1, 2, 3, 4, \cdots$$

To determine the fourth order moment,

$$R_{\phi\phi}(t) = E[\, b(u+s+t)b^*(u+t)b^*(u+s)b(u)\,] \tag{A.2}$$

we proceed to evaluate equation (A.1) for N=4 with

$$p_1 = 1, \quad p_2 = -1, \quad p_3 = -1, \quad p_4 = 1.$$

In vector notation this is

$$\underline{p}^T = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

Equation (A.2) is evaluated using equation (A.1) with appropriate substitutions for $\underline{p}$, N, M, and the random variable $\underline{z}$. We have

$$R_{\phi\phi}(t) = F_{\underline{z}}(j,-j,-j,\,j) \triangleq E[\exp(jz_1 - jz_2 - jz_3 + jz_4)] \tag{A.3a}$$

$$= \exp(\,-1/2(m_{11} + m_{22} + m_{33} + m_{44}) + (m_{12} + m_{13} + m_{24} + m_{34}) - (m_{14} + m_{23})\,)$$

In a similar manner the second order moment is

$$R(s) = F_{\underline{z}}(j,-j) \triangleq E[\exp(jz_1 - jz_2)] = \exp(-1/2(m_{11} + m_{22}) + m_{12}) \tag{A.3b}$$

other moments of $b(u) = \exp(jz(u))$ can generally be determined by the method just presented by assigning p=1 for b(.) and p=-1 for $b^*(.)$.

REFERENCES

[A.1] Snyder, Donald L., "Random Point Processes", John Wiley Sons, New York, 1975, Sic. 3.2, pp 130-137.

[A.2] Davenport, W. B. and Root, W. L., "Random Signals and Noise", McGraw-Hill, New York, 1958, sec 6-6 pp 107-108, sec 12-3 pp 257-259.

APPENDIX B

DERIVATION OF THE COVARIANCE MATRIX FOR COMBINED

WHITE PHASE AND RANDOM WALK PHASE INSTABILITY, $\underline{x} + \underline{y}$

The covariance matrix, M, of the zero mean random vector, $\underline{z}$, is defined as the expectation

$$M = E\left\{\underline{z}\,\underline{z}^{*T}\right\} = E\left\{(\underline{x} + \underline{y})(\underline{x} + \underline{y}^{*})^{*T}\right\} \qquad (B.1)$$

Since $\underline{x}$ and $\underline{y}$ are independent we have

$$M = E\left\{\underline{x}\,\underline{x}^{*T} + \underline{y}\,\underline{y}^{*T}\right\} = M_{\underline{x}} + M_{\underline{y}} \qquad (B.2)$$

For the Wiener component, $\underline{x}$, the covariance matrix is

$$M_{x} = E\left\{\underline{x}\,\underline{x}^{*T}\right\} = E\left\{\begin{bmatrix} x(u+t) \\ x(u) \end{bmatrix}\begin{bmatrix} x^{*}(u+t) & x^{*}(u) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} E\{x(u+t)\,x^{*}(u+t)\} & E\{x(u+t)\,x^{*}(u)\} \\ E\{x(u)\,x^{*}(u+t)\} & E\{x(u)\,x^{*}(u)\} \end{bmatrix}.$$

Carrying out the expectation operation we obtain

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$$
M_x = \begin{bmatrix} \mathrm{Var}\{x(u+t)\} & \mathrm{Cov}\{x(u+t)\,x^*(u)\} \\ \\ \mathrm{Cov}\{x(u)\,x^*(u+t)\} & \mathrm{Var}\{x(u)\} \end{bmatrix} \qquad (B.3)
$$

Substituting equation (2.4) of the text into the above we have

$$
M_x = \begin{bmatrix} (c_1 + |t|)\,\sigma_1^2 & c_1\,\sigma_1^2 \\ \\ c_1\,\sigma_1^2 & c_1\,\sigma_1^2 \end{bmatrix} \qquad (B.4)
$$

In a similar manner we can write the covariance matrix for the "near white" component, $\underline{y}(t)$.

$$
M_y = \begin{bmatrix} \mathrm{Var}\{y(u+t)\} & \mathrm{Cov}\{y(u+t)\,y^*(u)\} \\ \\ \mathrm{Cov}\{y(u)\,y^*(u+t)\} & \mathrm{Var}\{y(u)\} \end{bmatrix} \qquad (B.5)
$$

Substitution of equation (2.3) of the text into the above results in

$$
M_y = \begin{bmatrix} c_o\,\sigma_o^2 & (c_o - \dfrac{2}{T_o}|t|)\,\sigma_o^2 \\ \\ (c_o - \dfrac{2}{T_o}|t|)\,\sigma_o^2 & c_o\,\sigma_o^2 \end{bmatrix} \quad : \ |t| < \dfrac{T_o}{2} \qquad (B.6a)
$$

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$$M_y = \begin{bmatrix} c\,\sigma_o^2 & (c-1)\,\sigma_o^2 \\ (c-1)\,\sigma_o^2 & c\,\sigma_o^2 \end{bmatrix} \quad : \quad |t| > \frac{To}{2} \qquad (B.6b)$$

Substitution of equations (B.4) and (B.6) into equation (B.2) results in equation (2.9a) and (2.9b) of the text.

DERIVATION OF COVARIANCE MATRIX FOR PHASE INSTABILITY, $\underline{v}$, DUE TO

RANDOM WALK FREQUENCY AND LONG TERM FREQUENCY LINEAR DRIFT

C.0. Introduction

In this appendix we will derive the covariance matrix of the phase random process, $v(u)$, resulting from the integral of a random walk frequency process, $\dot{v}(u)$, i.e.,

$$v(u+t) = \int_{u}^{u+t} \dot{v}(y)\, dy \; + v(u).
\qquad (C.1)$$

For verification purposes, we will use three different approaches to arrive at the same result. We assume that the random walk frequency process, $\underline{\dot{v}}(u)$, is a zero mean Wiener process with covariance function

$$M._{\underline{v}} = E\{\underline{\dot{v}}\,\underline{\dot{v}}^{*T}\} = \begin{bmatrix} (c + |t|)\sigma^2 & c\,\sigma^2 \\[2mm] c\,\sigma^2 & c\,\sigma^2 \end{bmatrix} \qquad (C.2a)$$

where

$$\underline{\dot{v}} = \begin{bmatrix} \dot{v}(u+t) \\[2mm] \dot{v}(u) \end{bmatrix}, \qquad m._{\underline{v}} = E\{\underline{\dot{v}}\} = \underline{0},$$

σ_2^2 is the random walk frequency drift variance and $c_2 \sigma_2^2$ is the frequency offset variance, $\mathrm{var}\{\dot{v}(u)\}$, due to (past) long term linear drift [C.1], [C.2] at reference time, u. We assume that the phase random process, $v(u)$, also has a zero mean Gaussian random phase, independent of $\dot{v}(u)$, at reference time, u, with an arbitrary variance, $c_3 \sigma_2^2$. These conditions may be stated as follows:

$$\mathrm{Var}\{\dot{v}(u)\} = E\{\dot{v}(u)\ \dot{v}^*(u)\} = c_2 \sigma_2^2 \tag{C.2b}$$

$$\mathrm{Var}\{v(u)\} = E\{v(u)\ v^*(u)\} = c_3 \sigma_2^2 \tag{C.2c}$$

C.1. Approximation Function Approach

We shall first use a heuristic approach based on a direct application of the approxinating function [C.3] for determining the variance, $\mathrm{Var}\{v(u+t)\}$.

Suppose we assume the frequency takes on discrete random jumps, $\dot{v}(u_i)$: $u_o < u_1 < u_2 < .. < u_i < .. < u_n$ according to the random walk process. The frequency may also be written as a summation of independent terms

$$\dot{v}_i = \dot{v}_o + (\dot{v}_1 - \dot{v}_o) + (\dot{v}_2 - \dot{v}_1) + \ldots (\dot{v}_i - \dot{v}_{i-1})$$

$$\dot{v}_i = \dot{v}_o + \sum_{k=1}^{k=i} (\dot{v}_k - \dot{v}_{k-1}) \qquad (C.3)$$

where the parenthesized quantities are the k th independent increments and a notational change, $\dot{v}_i = \dot{v}(u_i)$ was made. $\dot{v}_o$ is the offset random frequency.

The phase random process, $v(u)$, is the integral of the above frequency process, the discrete form of which may be written:

$$v_n = v_o + \sum_{i=o}^{n} \dot{v}_i (u_{i+1} - u_i)$$

$$= v_o + \dot{v}_o (u_1 - u_o) + \dot{v}_1 (u_2 - u_1) + \dot{v}_2 (u_3 - u_2) + \ldots \dot{v}_n (u_{n+1} - u_n)$$

$$= v_o + \sum_{i=o}^{i=n} \dot{v}_i (u_{i+1} - u_i) \qquad (C.4)$$

where v_o is an arbitrary random phase. Equation (C.4) may also be written as follows:

$$\dot{v}_n = \dot{v}_o + \dot{v}_o(u_{n+1} - u_o) + (\dot{v}_1 - \dot{v}_o)(u_{n+1} - u_1) + .. (\dot{v}_n - \dot{v}_{n-1})(u_{n+1} - u_n)$$

$$= \dot{v}_o + \dot{v}_o(u_{n+1} - u_o) + \sum_{i=1}^{n}(\dot{v}_i - \dot{v}_{i-1})(u_{n+1} - u_i) \qquad (C.5)$$

Each term in the above expression is independent and of zero mean with boundary conditions specified in equations (C.2b) and (C.2c). Hence the variance may be written:

$$\text{var}\{v_n\} = c_3\sigma_2^2 + c_2\sigma_2^2(u_{n+1} - u_o)^2 + \sum_{i=1}^{n}\sigma_2^2(u_i - u_{i-1})(u_{n+1} - u_i)^2 \qquad (C.6)$$

We can evaluate the above summation in the limit.

$$\lim_{\substack{n \to \infty \\ \max\{u_i - u_{i-1}\} \to 0}} \sum_{i=1}^{n}\sigma_2^2(u_i - u_{i-1})(u_{n+1} - u_i)^2 = \sigma_2^2\int_{u_o}^{u_{n+1}}(u_{n+1} - y)^2\, dy$$

$$= \sigma_2^2\,\frac{(u_{n+1} - u_o)^3}{3}$$

Denoting points in time as

$$u_o = u \qquad \text{and} \qquad u_{n+1} = u + t$$

the limit of equation (C.6) can be written

$$\text{Var}\{v(u+t)\} = \lim_{\substack{n \to \infty \\ \max\{u_i - u_{i-1}\} \to 0}} \text{Var}\{v_n\} = c_3 \sigma_2^2 + c_2 \sigma_2^2 t + \sigma_2^2 \frac{t^3}{3}$$

$$= \sigma_2^2 \left(c_3 + c_2 t + \frac{t^3}{3} \right) \tag{C.7a}$$

We can also evaluate the variance for t=0

$$\text{Var}\{v(u)\} = c_3 \sigma_2^2 \tag{C.7b}$$

From the independence of the values, $\dot{v}_o$ and v_o, and the independence
of the increments, we can evaluate the covariance

$$\text{cov}\{v(u+t)v^*(u)\} = E\left\{\left(v_o + \int_u^{u+t} \dot{v}(y)dy\right)(v_o^*)\right\} = E\{v_o v_o^*\} = \text{var}\{v(u)\}$$

$$= c_3 \sigma_2^2 = \text{cov}\{v^*(u+t) \, v(u)\} \tag{C.7c}$$

Equation (C.7) provides the elements of the covariance matrix, $\underline{M}_v$,
for the non-stationary process

$$\underline{v} = \begin{bmatrix} v(u+t) \\ v(u) \end{bmatrix}$$

The covariance matrix is obtained by making the appropriate
substitutions into:

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$$M_{\underline{v}} = E\left\{ \begin{bmatrix} v(u+t) \\ v(u) \end{bmatrix} \begin{bmatrix} v^*(u+t) & v^*(u) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \text{var}\{v(u+t)\} & \text{cov}\{v(u+t)v^*(u)\} \\ \text{cov}\{v^*(u+t)v(u)\} & \text{var}\{v(u)\} \end{bmatrix} \qquad (C.8)$$

Substitution of equation (C.7) into (C.8) results in the desired expression which is used in equation (2.14) of the text..

C.2. Integration Formula Approach

Next we shall use an integration formula for determining the covariance matrix of the phase random process resulting from the random walk frequency process. We first determine the variance of the integration process, equation (C.1), at some instant of time, say (u+t).

From [C.4] (equations (5.3.1) and (5.3.4)), equation (C.1), the condition given in equation (C.2b) and (C.2c), and the independence assumption given above, we have for the integral of the random walk frequency process

$$I._{\underline{v}} \doteq \int_u^{u+t} \dot{v}(y)dy$$

and for the variance

$$\text{var}\{v(u+t)\} = E\left\{\left|\int_{u}^{u+t} \dot{v}(a)da + v(u)\right|^2\right\} = E\{|I_{.v}|^2\} + c_3 \sigma_2^2 \quad (C.9)$$

Evaluating the first term on the right we have

$$E\{|I_{.v}|^2\} = Q_1 = \int_{u}^{u+t} \int_{u}^{u+t} R_{.v}(p,q)\ dp\ dq \quad (C.10)$$

The auto-correlation function, $R_{.v}(p,q)$, denoted in equation (5.3.4) of [C.4] and in equation (C.10) above may be determined with the aid of equation (C.2a)

$$R_{.v}(p,q) = \sigma_2^2 c_2 + \sigma_2^2 \min\{p-u,q-u\} \qquad :p,q > u$$

Equation (C.10) becomes

$$E\{|I_{.v}|^2\} = \int_{u}^{u+t} \int_{u}^{u+t} [\sigma_2^2 c_2 + \sigma_2^2 \min\{p-u,q-u\}]\ dp\ dq$$

$$E\{|I_{.v}|^2\} = \sigma_2^2 \int_{u}^{u+t} dq \int_{u}^{u+t} [c_2 + \min\{p-u,q-u\}]\ dp$$

$$E\{|I_v|^2\} = \sigma_2^2 \left[\int_u^{u+t} c_2 t \, dq + \int_u^{u+t} dq \left[\int_u^q (p-u) \, dp + \int_q^{u+t} (q-u) \, dp \right] \right]$$

$$= \sigma_2^2 \left\{ c_2 t^2 + \int_u^{u+t} dq \left[\left|_u^q \frac{(p-u)^2}{2} + \right|_q^{u+t} (q-u)p \right] \right\}$$

$$= \sigma_2^2 \left[c_2 t^2 + \int_u^{u+t} dq \left[\frac{(q-u)^2}{2} + (q-u)(u+t-q) \right] \right]$$

$$= \sigma_2^2 \left[c_2 t^2 + \int_u^{u+t} \left[t(q-u) - \frac{(q-u)^2}{2} \right] dq \right]$$

$$= \sigma_2^2 \left[c_2 t^2 + \left|_u^{u+t} \left[\frac{t(q-u)^2}{2} - \frac{(q-u)^3}{6} \right] \right]$$

$$= \sigma_2^2 \left[c_2 t^2 + \left[\frac{t t^2}{2} - \frac{t^3}{6} \right] \right] = \sigma_2^2 (c_2 t^2 + \frac{t^3}{3}) \qquad (C.11)$$

Substition of equation (C.11) into equation (C.9) results in

$$\text{var}\{v(u+t)\} = \sigma_2^2 (c_3 + c_2 t^2 + \frac{t^3}{3}) \qquad (C.12a)$$

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This is the same expression, equation (C.7a), obtained using the approximation function in approach one above. It can be shown, using the same reasoning as presented in approach one that equation (C.7b) and (C.7c) still hold.

$$\text{var}\{v(u)\} = c_3 \sigma_2^2 \qquad \text{(C.12b)}$$

$$\text{cov}\{v(u+t)v^*(u)\} = c_3 \sigma_2^2 \qquad \text{(C.12c)}$$

C.3. State Space Approach

We now use a state variable approach for determining the variance, $\text{var}\{v(u+t)\}$.

The procedure we follow is outlined in [C.5] (section 6.3.1, Property 14, pp 533). It states:

The variance matrix of the state vector, $\underline{v}(t)$ of a system

$$\dot{\underline{v}}(t) = \underline{F}(t)\,\underline{v}(t) + \underline{G}(t)\,u(t) \qquad \text{(C.13a)}$$

satisfies the differential equation

$$\dot{\bigwedge}_{\underline{v}}(t) = \underline{F}(t)\bigwedge_{\underline{v}}(t) + \bigwedge_{\underline{v}}(t)\underline{F}^T(t) + \underline{G}(t)Q(t)\underline{G}^T(t) \qquad \text{(C.13b)}$$

with condition

$$\Lambda_{\underline{v}}(u) = E[\ \underline{v}(u)\ \underline{v}^{*T}(u)\ ] \qquad\qquad (C.13c)$$

Observe that $\Lambda_{\underline{v}}(t) = K_{\underline{v}}(t,t)$ is the covariance matrix of the vector $\underline{v}$.

To use equation (C.13) for the random walk frequency we set

$$\underline{v}(t) = \begin{bmatrix} v_1(t) \\[2mm] v_2(t) \end{bmatrix}, \qquad \underline{F}(t) = \begin{bmatrix} 0 & 1 \\[2mm] 0 & 0 \end{bmatrix},$$

$$\underline{G}(t) = \begin{bmatrix} 0 \\[2mm] \sigma_2 \end{bmatrix}, \qquad \Lambda_{\underline{v}}(t) = E[\ \underline{v}(t)\ \underline{v}^{*T}(t)]$$

where $v_1(u)$ denotes the phase random process, $v_2(u)$ denotes the frequency random process and $u(t)$ is a zero mean white noise process with variance, $Q(t) = 1$. This latter result, $Q(t)$ is obtained from [C.5] (Property 13, equation (257) and the result immediately following equation (254a)).

Substitution of the above into equation (C.13a) and (C.13b) gives us (C.14a) and (C.14b) respectively

$$\begin{bmatrix} \dot{v}_1(t) \\[3mm] \dot{v}_2(t) \end{bmatrix} = \begin{bmatrix} v_2(t) \\[3mm] \sigma_2\, u(t) \end{bmatrix} \qquad\qquad (C.14a)$$

$$\dot{\Lambda}_{\underline{v}} = \begin{bmatrix} \overline{\dot{v}_1^2} & \overline{\dot{v}_1 \dot{v}_2} \\ \overline{\dot{v}_1 \dot{v}_2} & \overline{\dot{v}_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{v_1^2} & \overline{v_1 v_2} \\ \overline{v_1 v_2} & \overline{v_2^2} \end{bmatrix} + \begin{bmatrix} \overline{v_1^2} & \overline{v_1 v_2} \\ \overline{v_1 v_2} & \overline{v_2^2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \sigma_2 \end{bmatrix} \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} 0 & \sigma_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\,\overline{v_1 v_2} & \overline{v_2^2} \\ \overline{v_2^2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \tag{C.14b}$$

where the complex notation, *, has been omitted because the process, $\underline{v}(t)$, is a phase random process which has been modeled as a real process throughout this dissertation. Dependence on time, $v = v(t)$, is implied, and notation, $E[v] = \overline{v}$, is adopted. v_1 denotes the phase random process and v_2 denotes the frequency random process, with conditions at time, u, taken from equations (C.2c) and (C.2b) respectively in conjunction with the independence relation between $v(u)$ and $\dot{v}(u)$. The conditions at time, u, equation (C.13c), may be stated

$$\Lambda_{\underline{v}}(u) = E[\,\underline{v}(u)\underline{v}^T(u)\,] = \begin{bmatrix} c_3 & 0 \\ 0 & c_2 \end{bmatrix} o_2^2 \tag{C.14c}$$

The variance matrix Λ is easily solved. From equation (C.14) we can collect three first order differential equations along with their boundary conditions.

Differential Equation	Boundary Condition
$\dot{\bar{v}}^2_1 = 2 \overline{v_1 v_2}$	$\bar{v}^2_1(u) = c_3 \sigma^2_2$: an arbitrary constant
$\overline{\dot{v}_1 v_2} = \overline{v_1 \dot{v}_2} = \bar{v}^2_2$	$\overline{v_1 v_2}(u) = 0$
$\dot{\bar{v}}^2_2 = \sigma^2_2$	$\bar{v}^2_2(u) = c_2 \sigma^2_2$: a function of u, long term drift offset frequency

Solving for $\bar{v}^2_2(t)$ first, we have

$$\bar{v}^2_2(t) = \int \dot{\bar{v}}^2_2(t)\, dt + k_1 = \int \sigma^2_2\, dt + k_1 = \sigma^2_2 t + k_1$$

Applying its boundary condition at time, u, we can solve for k_1,

$$\bar{v}^2_2(u) = \sigma^2_2 u + k_1 = c_2 \sigma^2_2 \implies k_1 = (c_2 - u)\sigma^2_2$$

Substituting for k_1, we obtain the solution for $\bar{v}_2(t)$,

$$\bar{v}^2_2(t) = \sigma^2_2 (t + c_2 - u)$$

Next, we solve for the quantity, $\overline{v_1 v_2}(t)$.

$$\overline{v_1 v_2}(t) = \int \overline{\dot{v}_1 v_2}(t)\, dt + k_2$$

$$= \int \overline{v_2^2}(t)\, dt + k_2 = \sigma_2^2 \left(\frac{t^2}{2} + [c_2 - u]\, t \right) + k_2$$

Applying its boundary condition at time, u, we can solve for k_2,

$$\overline{v_1 v_2}(u) = \sigma_2^2 \left(c_2 u - \frac{u^2}{2} \right) + k_2 = 0 \implies k_2 = \sigma_2^2 \left(\frac{u^2}{2} - c_2 u \right)$$

Substituting for k_2, we obtain the solution for $\overline{v_1 v_2}(t)$

$$\overline{v_1 v_2}(t) = \sigma_2^2 \left(\frac{[t-u]^2}{2} + c_2\, [t-u] \right)$$

Finally, we solve for the quantity, $\overline{v_1^2}(t)$

$$\overline{v_1^2}(t) = \int \overline{\dot{v}_1^2}(t)\, dt + k_3$$

$$\overline{v_1^2}(t) = 2 \int \overline{v_1 v_2}\, dt + k_3 = 2\sigma_2^2 \left(\frac{[t-u]^3}{6} + \frac{c_2\, [t-u]^2}{2} \right) + k_3$$

Applying the boundary condition at time, u, we can solve for k_3

$$\overline{v_1^2}(u) = k_3 = c_3 \sigma_2^2 \implies k_3 = c_3 \sigma_2^2$$

Substituting for k_3, we obtain the solution for $\overline{v_1^2}(t)$

$$\overline{v}_1^2(t) = \sigma_2^2 \left(\frac{[t-u]^3}{3} + c_2[t-u]^2 + c_3 \right)$$

The variance of the phase random process, $v_1(t)$, is obtained from this last expression. After a change in variables, $t' = (t-u)$, we have

$$\text{var}\{v_1(u+t')\} = E\{v_1^2(u+t')\} = \overline{v}_1^2(u+t') = \sigma_2^2 \left(c_3 + c_2 t' + \frac{t'^3}{3} \right) \qquad \text{(C.15a)}$$

This is the same expression obtained in equation (C.7a) of approach one and equation (C.12a) of approach two.

Using the independence assumption we have

$$\text{var}\{v(u)\} = c_3 \sigma_2^2 \qquad \text{(C.15b)}$$

$$\text{cov}\{v(u+t)v^*(u)\} = c_3 \sigma_2^2 \qquad \text{(C.15c)}$$

which are the desired results.

C.4. Summary of Approaches

From approach one, two, or three above, we obtain the covariance function, M_v of the phase random process, $v(u)$, resulting from the frequency random walk process, $\dot{v}(u)$;

$$M_v = E\{\,\underline{v}\,\underline{v}^{*T}\,\} = E\left\{ \begin{bmatrix} v(u+t) \\ v(u) \end{bmatrix} \begin{bmatrix} v^*(u+t) & v^*(u) \end{bmatrix} \right\}$$

$$= \sigma_2^2 \begin{bmatrix} c_3 + c_2\, t + \dfrac{t^3}{3} & c_3 \\[2ex] c_3 & c_3 \end{bmatrix} \qquad\qquad (C.16)$$

This result is used in the text, cf. section II.

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Determination of Second and Fourth Order Moments

of the RF Signal Process

0.0. Introduction

There are perhaps many approaches that can be used for determining
the higher order moments of oscillator rf signal process due to white
phase, $y(u)$, random walk phase, $x(u)$, and random walk frequency, $v(u)$.
In this appendix, we use two approaches, the characteristic function and
a method involving independent increments.

D.1. Characteristic Function

D.1.1. Near White Phase, $y(u)$

We determine the higher order moments of $b(u)$: $t \gg s$, for $y(u)$
Gaussian distributed with the aid of figure D.1 and the characteristic
function method [D.1] (chapter 4 and 8), [D.2] presented in Appendix A.

We define/construct the four element vector

$$\underline{y}_4^{\,T} \triangleq [\; y(u+s+t) \quad y(u+t) \quad y(u+s) \quad y(u) \;]$$

Using figure D.1 we can easily write the covariance matrix for

$$M_{\underline{y}4} = \frac{\sigma_o^2}{4} \begin{bmatrix} c_o & c_o - 2s/T_o & c_o - 1 & c_o - 1 \\ c_o - 2s/T_o & c_o & c_o - 1 & c_o - 1 \\ c_o - 1 & c_o - 1 & c_o & c_o - 2s/T_o \\ c_o - 1 & c_o - 1 & c_o - 2s/T_o & c_o \end{bmatrix} \quad \begin{array}{l} : \; s < \dfrac{T_o}{2} \\[1em] : \; t > \dfrac{T_o}{2} + s \end{array}$$

$$\text{(D.1)}$$

$$M_{\underline{y}4} = \frac{\sigma_o^2}{4} \begin{bmatrix} c_o & c_o - 1 & c_o - 1 & c_o - 1 \\ c_o - 1 & c_o & c_o - 1 & c_o - 1 \\ c_o - 1 & c_o - 1 & c_o & c_o - 1 \\ c_o - 1 & c_o - 1 & c_o - 1 & c_o \end{bmatrix} \quad \begin{array}{l} : \; s > \dfrac{T_o}{2} \\[1em] : \; t > \dfrac{T_o}{2} + s \end{array}$$

We will denote the elements of the matrix, $M_{\underline{y}4}$ by m_{ij}. Substitution of equation (D.1) into equation (A.3) leads to the fourth order moment for the rf waveform, $b_y(u)$.

$$R_{\phi\phi y}(t) = \exp(-4s\sigma_o^2/T_o) \qquad : \; s < T_o/2 < t-s$$

$$\text{(D.2a)}$$

$$R_{\phi\phi y}(t) = \exp(-2\sigma_o^2) \qquad : \; T_o/2 < s < t - T_o/2$$

In much the same manner the second order moment of the rf waveform is

$$R_y(s) = \exp(-2s\sigma_o^2/T_o) \qquad : \; s < T_o/2 < t-s$$

$$\text{(D.2b)}$$

$$R_y(s) = \exp(-\sigma_o^2) \qquad : \; T_o/2 < s < t - T_o/2$$

The fourth and second order moments for the white phase component are given by equations (D.2a) and (D.2b) respectively.

D.1.2. <u>Random</u> <u>Walk</u> <u>Phase,</u> <u>x(u)</u>

We revisit the random walk phase process. Our vector random process is

$$\underline{x}_4^T \triangleq [\; x(u{+}s{+}t) \quad x(u{+}t) \quad x(u{+}s) \quad x(u) \;]$$

With the aid of figure D.2 we write the covariance matrix

$$
M_{\underline{x}4} = \sigma_1^2
\begin{bmatrix}
c_1 + s + t & c_1 + t & c_1 + s & c_1 \\
c_1 + t & c_1 + t & c_1 + t & c_1 \\
c_1 + s & c_1 + t & c_1 + s & c_1 \\
c_1 & c_1 & c_1 & c_1
\end{bmatrix}
\quad : \; |t| < |s|
$$

$$(D.3)$$

$$
M_{\underline{x}4} = \sigma_1^2
\begin{bmatrix}
c_1 + s + t & c_1 + t & c_1 + s & c_1 \\
c_1 + t & c_1 + t & c_1 + s & c_1 \\
c_1 + s & c_1 + s & c_1 + s & c_1 \\
c_1 & c_1 & c_1 & c_1
\end{bmatrix}
\quad : \; |s| < |t|
$$

Substitution of equation (D.3) into equation (A.3) leads to the fourth order moment of the rf signal when the phase is a random walk Gaussian process.
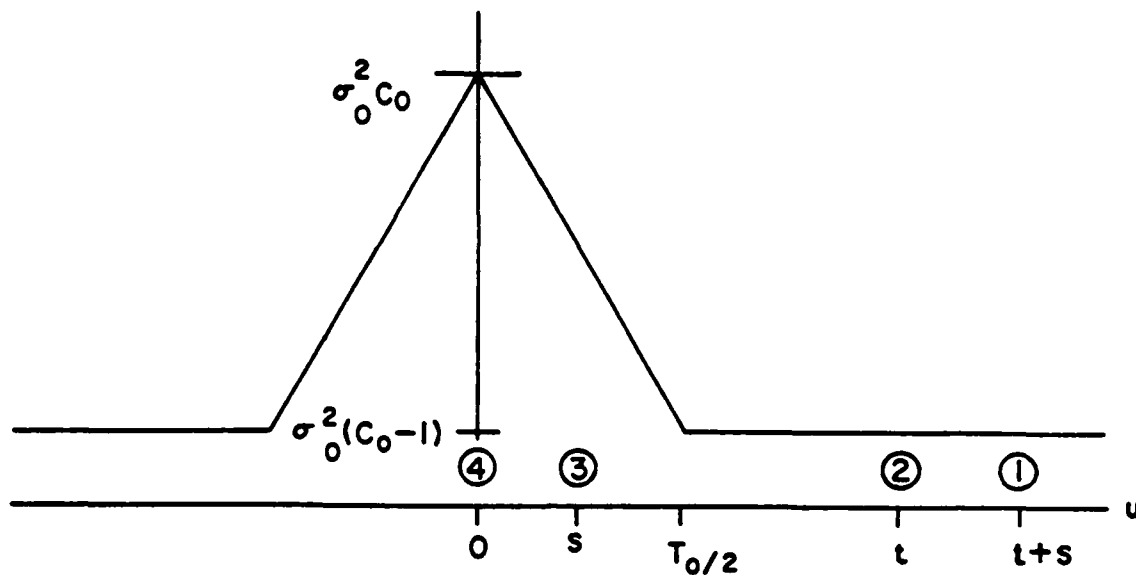
FIGURE D.1   AUTO-CORRELATION FUNCTION OF NEAR WHITE PHASE, $y(u)$
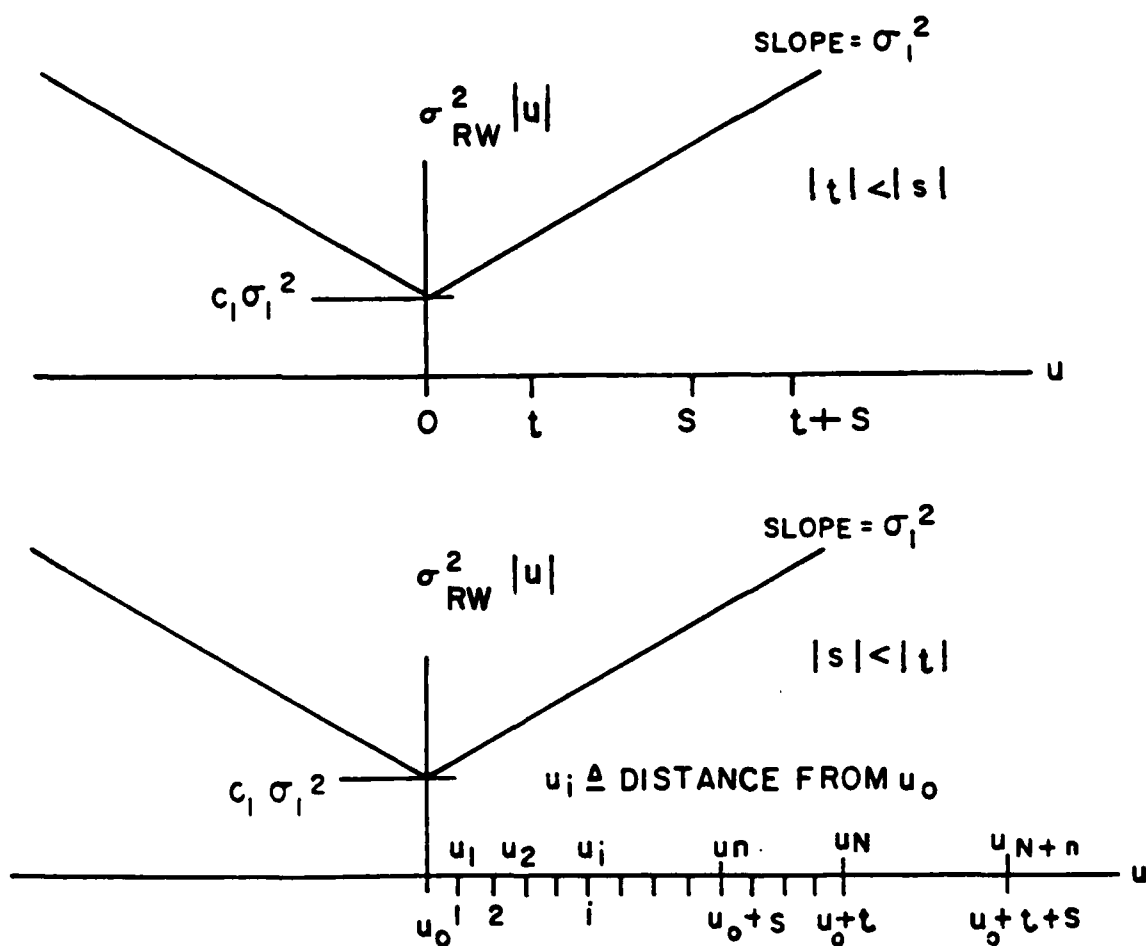
FIGURE D.2   VARIANCE OF RANDOM WALK PROCESS, $x(u)$

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$$R_{\phi\phi x}(t) = \exp(-\sigma_1^2 |t|) \qquad : \quad |t| < |s| \tag{D.4a}$$

$$R_{\phi\phi x}(t) = \exp(-\sigma_1^2 |s|) \qquad : \quad |s| < |t|$$

This is in agreement with equation (D.9) which will be derived as the product of two independent increments. The second order moment is easily obtained using the same procedure,

$$R_x(s) = \exp(-\sigma_1^2 \frac{|s|}{2}) \tag{D.4b}$$

D.1.3. Random Walk Frequency, $\dot{v}(u)$

Four Element Vector

In order to determine the fourth order moment of b(u) when the phase random process arises from a Gaussian random walk frequency component, $\dot{v}(u)$, we must determine the covariance matrix of

$$\underline{v}_4^T \doteq [\ v(u+s+t)\quad v(u+t)\quad v(u+s)\quad v(u)\ ]$$

as we did for the white phase and random walk phase processes above. The approximating function, [D.3], (chapter 7, sec.5), a discrete approximation of the continuous function, can be made to provide a systematic procedure for determining the elements of the covariance matrix, $M_{\underline{v}4}$. Using figure D.2 for the case, $|s| < |t|$

$$\dot{v}_m = \dot{v}_o + \ddot{v}_o(u_m - u_o) + \sum_{i=1}^{m}(\dot{v}_i - \dot{v}_{i-1})(u_m - u_{i-1}) \qquad (D.5)$$

where $m = o,\ n,\ N,\ n+N$ is the number of discrete steps constituting

time $u = u_o,\ u_o + s,\ u_o + t,\ u_o + t + s$ respectively, see figure D.2.

For $\dot{v}_o$, $\ddot{v}_o$, and $(\dot{v}_i - \dot{v}_{i-1})$ zero mean independent and real, we may write

the covariance matrix elements of $\underset{-4}{v}$.

The diagonal terms can be evaluated as follows.

$$\mathrm{Var}\{v_m\} \triangleq E\{v_m^2\}$$

Substitution of equation (D.5) into the above we have

$$\mathrm{Var}\{v_m\} = E\{\dot{v}_o^2\} + E\{\ddot{v}_o^2(u_m - u_o)^2\} + \sum_{i=1}^{m} E\{(\dot{v}_i - \dot{v}_{i-1})^2(u_m - u_{i-1})^2\}$$

Evaluating the expectation and using the property of the drift variance
and stationary independent increments associated with the Brownian
motion process under the summation we have

$$\mathrm{Var}\{v_m\} = c_3\sigma_2^2 + c_2\sigma_2^2(u_m - u_o)^2 + \sum_{i=1}^{m}\sigma_2^2(u_m - u_{i-1})(u_i - u_{i-1})^2$$

which in the limit leads to

$$\text{Var}\{v(u_o + t)\} = \lim_{\substack{o < m \to \infty \\ \max\{u_i - u_{i-1}\} \to o}} \left[\text{Var}\{v_m\} \right]$$

$$= c_3 \sigma_2^2 + c_2 \sigma_2^2 t^2 + \int_{u_o}^{u_o + t} \sigma_2^2 (u_o + t - x)\, dx = \sigma_2^2 (c_3 + c_2 t^2 + \frac{t^2}{3}) \qquad \text{(D.6a)}$$

where x is the variable of intergration corresponding to the dummy

index, i, and $u_o + t$ is designated as time, u_m.

The off-diagonal terms are similarly evaluated. Let

$m_1 < m_2$ denote time, $u_{m1} < u_{m2}$: where m1, m2 are the number of discrete

increments along the time axis, u, and u_{m1}, u_{m2} correspond to shifts

from u_o along u.

Next we evaluate the off diagonal elements of the covariance matrix.

$$\text{Cov}\{v_{m1} v_{m2}\} \triangleq E\{v_{m1} v_{m2}\}$$

Substitution of equation (D.5) into the above gives

$$\text{Cov}\{v_{m1} v_{m2}\} = E\{v_o^2\} + E\{\dot{v}_o (u_o + u_{m1} - u_o)(u_o + u_{m2} - u_o)\}$$

$$+ E\left\{ \left[\sum_{i=1}^{m1} (\dot{v}_i - \dot{v}_{i-1})(u_{m1} - u_{i-1}) \sum_{j=1}^{m2} (\dot{v}_j - \dot{v}_{j-1})(u_{m2} - u_{j-1}) \right] \right\}$$

where m1, and m2 denotes any of the subscripts, m = o, n, N, n+N.

From the stationary independent increments

$$\text{Cov}\{v_{m1}\, v_{m2}\} = c_3\sigma^2 + c_2\sigma^2 u_{m1} u_{m2} + \sum_{i=1}^{m1} E\{(v_i - v_{i-1})(u_{m1} - u_{i-1})(u_{m2} - u_{i-1})\}$$

$$= \sigma_2^2(c_3 + c_2 u_{m1} u_{m2}) + \sum_{i=1}^{m1} \sigma_2^2 (u_{m1} - u_{i-1})(u_{m2} - u_{i-1})(u_i - u_{i-1})$$

Taking the limit as the number of discrete steps, for m_1 and m_2, each approach infinity we have

$$\text{Cov}\{v(u_o + u_{m1})v(u_o + u_{m2})\} = \lim_{\substack{m1 < m2 \to \infty \\ \max\{u_i - u_{i-1}\} \to 0}} \left[\text{Cov}\{v_{m1}\, v_{m2}\} \right]$$

$$= \sigma_2^2(c_3 + c_2 u_{m1} u_{m2}) + \int_{u_o}^{u_o + u_{m1}} \sigma_2^2 (u_o + u_{m1} - x)(u_o + u_{m2} - x)\, dx$$

$$= \sigma_2^2 \left(c_3 + c_2 u_{m1} u_{m2} + u_{m2}\frac{u_{m1}^2}{2} - \frac{u_{m1}^3}{6} \right) \qquad (D.6b)$$

Equation (D.6) can be used to determine the covariance matrix of the vector, $\underline{v}_4$, equation (D.6a) for the diagonal elements and equation (D.6b) for the off-diagonal elements. For example, substituting t for u_{m1} and t+s for u_{m2} we can determine the covariance

matrix element, m_{12}; or substitutings for u_{m1} and t for u_{m2} we determine the element m_{23}. Using the vector definition from above

$$\underline{v}_4^T = [\ v(u+s+t) \quad v(u+t) \quad v(u+s) \quad v(u)\ ]$$

the covariance matrix element, m_{23} can be determined from equation (D.6b)

$$m_{23} = \text{Cov}\{v(u_o + u_{m1})v(u_o + u_{m2})\}\Big|_{u_{m1} = s,\ u_{m2} = t}$$

$$= \sigma^2(c_2 + c_3 u_{m1} u_{m2} + u_{m2}\frac{u_{m1}^2}{2} - \frac{u_{m1}^3}{6})\Big|_{u_{m1} = s,\ u_{m2} = t}$$

$$= \sigma^2(c_2 + c_3 st + \frac{s^2}{2} t - \frac{s^3}{6})$$

Substitution of the appropriate variables for u_{m1} and u_{m2} into equation (D.6) for the four element vector, the covariance matrix elements are determined as a function of s and t. Then substituting these matrix elements into the characteristic function of equation (A.3a), the fourth order moment of the rf waveform, $b(u)$, is found.

$$R_{\phi\phi b}(t) = \exp(-\sigma^2 \frac{s^2}{6}[3t - s]) \qquad : \ |s| < |t| \qquad (D.7a)$$

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In a similar fashion the second order moment of $b(u)$ is determined using equation (A.3b).

$$R_b(s) = \exp(-\sigma_2 \frac{s^2}{6} [3c_2 + s]) \qquad (D.7b)$$

Next we use an alternative approach for determining the phase covariance matrix.

D.2. Independent Increments

This approach makes use of the independent increments which constitute the random walk processes involving phase and frequency. It has the advantage of decomposing a four by four element matrix and corresponding four element vector characteristic function into a two by two matrix and two element vector characteristic function. We will find that the procedure greatly simplifies determination of the covariance matrix for the random walk phase and frequency processes as well as serving as a check against the method for the four element characteristic function just presented.

D.2.1. Stationary Increments (Brownian Motion/Random Walk Phase)

For any random walk phase process the fourth order moment required in the theorems of chapter III can be evaluated as follows.

$$R_{\phi\phi}(t) = E[b(u+s+t) b^*(u+t) b^*(u+s) b(u)] \qquad (D.8)$$

$$= E[\exp\{ jx(u+s+t) - jx(u+t) - jx(u+s) + jx(u) \}]$$

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By examining the exponent and making use of the stationary independent increments the phase term, x(u), in the exponent can be rewritten as the sum of two independent random variables,

$$[x(u+s+t) - x(u+t)] + [x(u) - x(u+s)] \qquad : \ |s| < |t|$$

$$x(u+s+t) - x(u+s)] + [x(u) - x(u+t)] \qquad : \ |s| > |t|$$

This is illustrated in figure D.3.

The fourth order moment for the random walk Gaussian distributed phase component only is

$$R_{\phi\phi b}(t) = \exp(-\frac{\sigma_1^2}{2}|t|) \exp(-\frac{\sigma_1^2}{2}|t|) = \exp(-\sigma_1^2|t|) \qquad : \ |t| < |s|$$

$$(D.9)$$

$$R_{\phi\phi b}(t) = \exp(-\frac{\sigma_1^2}{2}|s|) \exp(-\frac{\sigma_1^2}{2}|s|) = \exp(-\sigma_1^2|s|) \qquad : \ |s| < |t|$$

where $|s|$, $|t|$ is the appropriate first order increment which emerged from Appendix B. The result is identical to the fourth order moment given in equation (D.4a), where the characteristic function method was used.

D.2.2. Two Element Increment Vector for Random Walk Frequency

Let us refer back to figure D.2. We define the two element vector

$$\underline{v}_2^T \triangleq [v(u+s+t) - v(u+t) \qquad v(u+s) - v(u)]$$

FOR CORRELATION FUNCTIONS HAVING FACTORS E.O.(EXPONENTIAL ORDER)<-0
e.g. RANDOM WALK PHASE & RANDOM WALK FREQUENCY

$t<-S$

U_0+t U_0+t+S U_0 U_0+S

$$R_{\phi\phi}(t)=e^{-\sigma_1^2\,|S|}$$

$-S<t<0$

U_0+t+S

U_0+t U_0 U_0+S

$$R_{\phi\phi}(t)=e^{-\sigma_1^2\,|t|}$$

$0<t<S$

U_0+t+S

U_0 U_0+t U_0+S

$$R_{\phi\phi}(t)=e^{-\sigma_1^2\,|t|}$$

$S<t$

U_0+t

U_0 U_0+S U_0+t+S
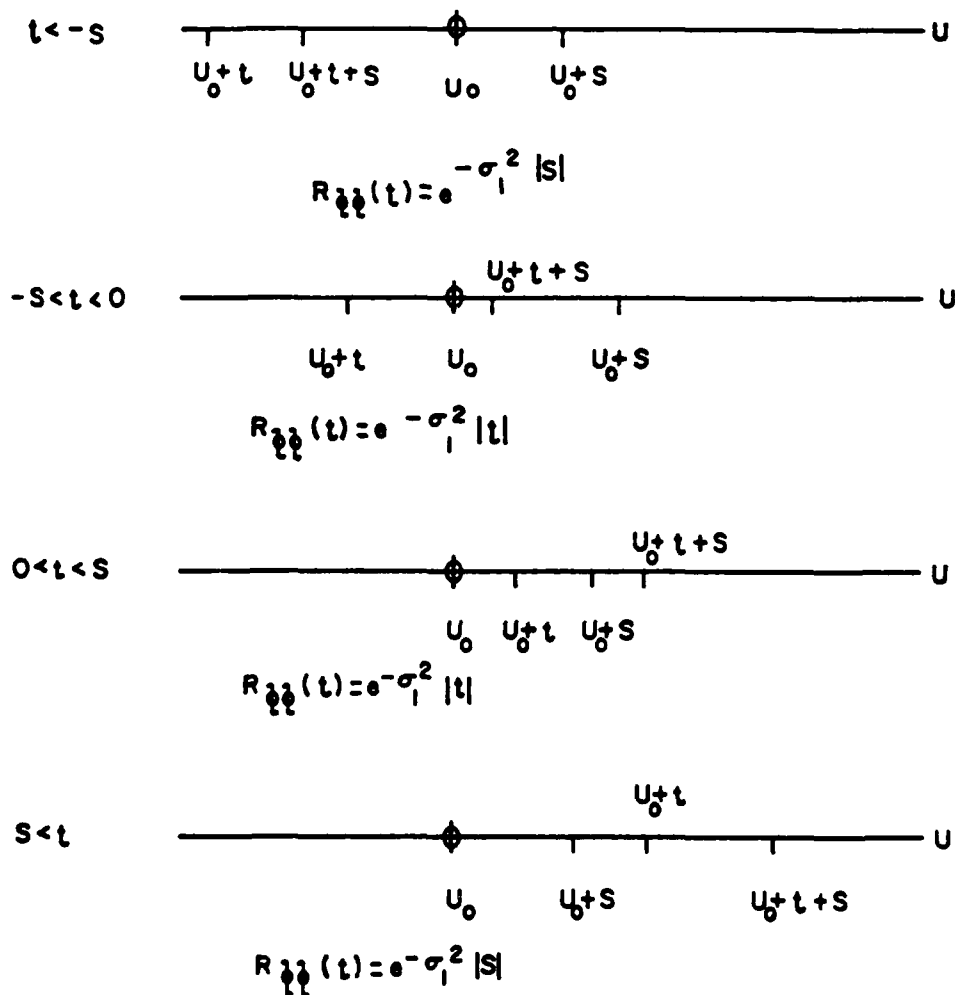
$$R_{\phi\phi}(t)=e^{-\sigma_1^2\,|S|}$$

FIGURE D.3   DOMAIN FOR FOURTH ORDER MOMENTS OF WAVEFORM, b(u)

Again, in our approximating function approach, we set a number of

discrete increments to m1 and m2 where m1 < m2 and substitute

u_{m1} for the shift, t and u_{m2} for the shift, t+s, and u_{m2-m1} for the

shift, s. The shift is from u_o along the time axis, u. The diagonal

terms of the covariance matrix of the vector $\underline{v}_2$ can be written

$$\text{Var}\{v_{m2} - v_{m1}\} \triangleq E\{(v_{m2} - v_{m1})^2\}$$

Substitution of the approximating model, equation (D.5), into the above

gives us, after some manipulation,

$$\text{Var}\{v_{m2} - v_{m1}\} = E\{\dot{v}_o^2 (u_{m2} - u_{m1})^2\}$$

$$+ E\left\{ \left[\sum_{i=1}^{m1} (\dot{v}_i - \dot{v}_{i-1})(u_{m2} - u_{m1}) \right]^2 + \left[\sum_{i=m1+1}^{m2} (\dot{v}_i - \dot{v}_{i-1})(u_{m2} - u_{i-1}) \right]^2 \right\}$$

Taking the limit we have

$$\text{Var}\{v(u + u_{m2}) - v(u + u_{m1})\} = \lim_{\substack{m1, m2 \to \infty \\ \max\{u_i - u_{i-1}\} \to 0}} \left[\text{Var}\{v_{m2} - v_{m1}\} \right]$$

$$= c_2 \sigma_2^2 (u_{m2} - u_{m1})^2 + \int_{u_o}^{u_o + u_{m1}} \sigma_2^2 (u_{m2} - u_{m1}) \, dx + \int_{u_o + u_{m1}}^{u_o + u_{m2}} \sigma_2^2 (u_o + u_{m2} - x) \, dx$$

The desired result is

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$$\text{Var}\{ v(u + u_{m2}) - v(u + u_{m1}) \}$$

$$= \sigma_2^2 \left[c_2 (u_{m2} - u_{m1})^2 + u_{m1}(u_{m2} - u_{m1}) + \frac{(u_{m2} - u_{m1})^3}{3} \right] \quad \text{(D.10a)}$$

The off diagonal terms may be found by designating the second element in the vector v_2 by the substitution, u_{m2-m1} for $u + s_o$ where

$$0 < u_{m2} - u_{m1} < u_{m1} < u_{m2}.$$

Then it follows

$$\text{Cov}\{(v_{m2} - v_{m1})(v_{m2-m1} - v_o)\} \triangleq E\{(v_{m2} - v_{m1})(v_{m2-m1} - v_o)\}$$

$$= E\{\dot{v}_o^2 (u_{m2} - u_{m1})(u_{m2-m1} - u_o)\} + E\left[\sum_{i=1}^{m2-m1} (\dot{v}_i - \dot{v}_{i-1})^2 (u_{m2} - u_{m1})(u_{m2-m1} - u_o) \right]$$

Taking the limit we have

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$$\mathrm{Cov}\{[v(u_{m2}+u) - v(u + u_{m1})][v(u + u_{m2-m1}) - v(u)]\}$$

$$= \lim_{\substack{m2-m1,\ m1 \to \infty \\ \max\{u_i - u_{i-1}\} \to 0}} = \sigma_2^2\, c(u_{m2} - u_{m1})(u_{m2-m1} - u_o)$$

$$+ \int_{u_o}^{u_o + u_{m2-m1}} \sigma_2^2 (u_{m2} - u_{m1})(u_o + u_{m2-m1} - x)\, dx \qquad (D.10b)$$

Making the appropriate substitutions for u_{m1} and u_{m2} into equation (D.10) for the two element increment vector, the covariance matrix elements are determined as a function of time shifts s and t. Substitution of equation (D.10) into equation (A.3b) results in the fourth order moment of the rf signal, b(u). Note that it is identical to a previous result found in equation (D.7a),

$$R_{oob}(t) = \exp\left[- \sigma_2^2\, \frac{s^2}{6}\, (3t - s)\right] \qquad : |s| < |t| \qquad (D.7a)$$

The second order moment is determined by similar methods using a one dimensional vector for v(u), i.e. a scaler

$$R_b(s) = \exp\left[- \sigma_2^2\, \frac{s^2}{6}\, \left(3c + \frac{s}{2}\right)\right] \qquad (D.7b)$$

D.3. Summary

Using characteristic functions a systematic procedure was obtained [D.2] for evaluating moments and auto-correlation functions of the rf signal, b(u), as required in the theorems of chapter III. This was demonstrated for each phase instability process treated here; the white and random walk phase and frequency including initial starting phase.

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